ACT-07-12, MIFPA-12-15

Momentum Modes of $M5$-branes in a $2d$ Space

Shan Hu [1,a], **and Dimitri Nanopoulos** [1,2,3,b]

[1] *George P. and Cynthia W.Mitchell Institute for Fundamental Physics, Texas A&M University, College Station, TX 77843, USA*

[2] *Astroparticle physics Group, Houston Advanced Research Center (HARC), Mitchell Campus, Woodlands, TX 77381, USA*

[3] *Academy of Athens, Division of Nature Sciences, 28 panepistimiou Avenue, Athens 10679, Greece*

[a] `hushan@physics.tamu.edu`, [b] `dimitri@physics.tamu.edu`

Abstract

We study $M5$ branes by considering the selfdual strings parallel to a plane. With the internal oscillation frozen, each selfdual string gives a $5d$ SYM field. All selfdual strings together give a $6d$ field with 5 scalars, 3 gauge degrees of freedom and 8 fermionic degrees of freedom in adjoint representation of $U(N)$. Selfdual strings with the same orientation have the SYM-type interaction. For selfdual strings with the different orientations, which could also be taken as the unparallel momentum modes of the $6d$ field on that plane or the (p,q) (r,s) strings on $D3$ with $(p,q) \neq (r,s)$, the $[i,j] + [j,k] \to [i,k]$ relation is not valid, so the coupling cannot be written in terms of the standard $N \times N$ matrix multiplication. 3-string junction, which is the bound state of the unparallel $[i,j]$ $[j,k]$ selfdual strings, may play a role here.

Keywords: Field Theories in Higher Dimensions, Brane Dynamics in Gauge Theories, M-Theory

1 Introduction

The effective theory on $M5$ branes is special in that the basic excitations, the selfdual strings, are $1d$ objects other than the $0d$ objects, like those on D branes or $M2$ branes [1, 2]. If the selfdual strings can be closed and can shrink to point like the fundamental strings, then we will still have a theory with the semi-point-like excitations. For selfdual strings without the charge, this is indeed the case. In abelian theory, the quantization of the point-like $M2$ confined to $M5$ brane gives the $(2,0)$ tensor multiplet [3, 4, 5]. Moreover, the basic excitations on $(2,0)$ little string theory [6] living on N coincident type IIA $NS5$ branes are closed fundamental strings, which are also the closed selfdual strings coming from $M2$ wrapping the M theory circle intersecting $NS5$ along a closed curve.

There is no evidence showing that for selfdual strings carrying charge, the situation is the same. Consider $D4$ branes, which are $M5$ branes compactified on x_5. The $[i,j]$ monopole string on $D4$ can carry the $D0$ charge. The closed $[i,j]$ monopole string with the vanishing length carrying $D0$ will appear as the point-like instanton with charge $[i,j]$. However, the classical instanton solution with charge $[i,j]$ is associated with the $[i,j]$ monopole string extending along a straight line, while the point-like[1] $1/2$ BPS instanton solutions are always chargeless. The closed monopole string with no $D0$ charge is the selfdual string with the winding number and the momentum both zero along x_5, which will give a $5d$ massless SYM field in adjoint representation of $U(N)$, in addition to the original $5d$ SYM field coming from the selfdual strings winding x_5 once. The SYM field like this is always massless even if the $D4$ branes are separated from each other, so it will dominate at the Coulomb branch. However, on $D4$, no such field exists. We do not get the clue for the existence of the closed charged selfdual strings. Actually, when the charged selfdual string becomes curved, different parts of it may exert force to each other, so it cannot vibrate freely and cannot be closed as the chargeless strings do.

Then we have to incorporate the $1d$ object in a $6d$ field theory. In this paper, we will take the $[i,j]$ tensionless selfdual string extending along, for example, the x_5 direction, as the point-like $6d$ excitation, which is in the position eigenstate in 1234 space but in the $P_5 = 0$ momentum eigenstate in x_5. If so, selfdual strings extending along the same direction cannot give the complete Hilbert space for the 6d particle. To get the full Hilbert space, we need to consider selfdual strings with the orientations covering all directions in a plane. The superposition of the selfdual strings parallel to a plane can give the 6d point-like excitations localized in 12345 space, but it seems that somehow, the position representation is not the suitable one, since it is the $[i,j]$ selfdual strings other than the $[i,j]$ point-like excitations that naturally exist. One plane is already enough to define a $6d$ field theory, so different planes may give the U-dual versions of the same $6d$ theory. This is quite similar with the $\mathcal{N} = 4$ SYM theory, for which, one (p,q) string defines a $4d$ field theory, while the rest (p,q) strings give the S-dual $4d$ theories.

[1] By point-like, we mean the instanton solution is localized in R^4, centered around a point.

Theories with the line-like excitations are intrinsically different from those with the point-like excitations. If the excitations are line-like, a reduction on x_5 will give the selfdual strings extending along x_5, while a further reduction on x_4 will make $P_4 = 0$. The first reduction selects a particular selfdual string; the second one is just the ordinary reduction in local field theories. With 4 and 5 switched, we will get the selfdual strings extending along x_4 with $P_5 = 0$, which is S-dual to the selfdual strings extending along x_5 with $P_4 = 0$. On the other hand, if the excitations are point-like, both sequences will give the point-like selfdual strings with $P_4 = P_5 = 0$. In [7, 8], Witten has shown that due to the conformal symmetry, the 45 and 54 reductions of the $6d$ $(2,0)$ theory will give two S-dual $4d$ SYM theories other than one $4d$ theory, which strongly indicates that the basic excitations on $M5$ cannot be point-like.

Recall that in [9], the equations of motion for the 3-algebra valued $(2,0)$ tensor multiplet contain a constant vector field C_μ. A given C_μ will reduce the dynamics from $6d$ to $5d$. However, if C_μ covers all directions in a plane, we will get a set of θ-parameterized $5d$ SYM theories[2], which is equivalent to a $6d$ theory with 5 scalars, 3 gauge degrees of freedom and 8 fermionic degrees of freedom. Each $5d$ SYM theory has the $5d$ vector multiplet in adjoint representation of $U(N)$, arising from the quantization of the open $[i,j]$ $M2$ intersecting $M5$ along the $C_\mu(\theta)$ direction. The oscillation along $C_\mu(\theta)$ is frozen, so the spectrum is the same as that from the quantization of the open string. For the given $6d$ $(2,0)$ tensor multiplet field configuration, each $5d$ SYM field comes from the reduction of the $6d$ field along $C_\mu(\theta)$. This is actually a special kind of KK compactification, using the polar coordinate other than the rectangular coordinate.

Suppose the selfdual string orientations are restricted in 45 plane, then the common eigenstates of $[\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{P}_4, \hat{P}_5]$ could be selected as the bases to generate the Hilbert space. In this respect, it is convenient to consider the KK mode of the $6d$ $(2,0)$ theory with x_4 and x_5 compactified to circles with the radii R_4 and R_5. $M5$ branes with the longitudinal x_4 and x_5 compactified is dual to the $D3$ branes with the transverse x'^{45} compactified. The vacuum expectation values of the 2-form field on $x_4 \times x_5$ is converted to the transverse positions of the $D3$ branes on x'^{45}. The duality differs from the T-duality in that two longitudinal dimensions are converted to one transverse dimension so the total dimensions are reduced from 11 to 10. The $(n/R_4, m/R_5)$ momentum mode of the $6d$ theory is dual to the (p,q) string winding x'^{45} k times, with $n = kp$, $m = -kq$, p and q co-prime. The $[i,j]$ (p,q) strings form the adjoint representation of $U(N)$. Correspondingly, the momentum modes as well as the original $6d$ field are also in the $U(N)$ adjoint representation.

The $(n/R_4, m/R_5)$ momentum modes are in $4d$ vector multiplet V_4, which, when combine together, give the $6d$ tensor multiplet T_6 in $U(N)$ adjoint representation. Although the $6d$ tensor multiplet is in the adjoint representation, the coupling involving more than two fields cannot be realized as the standard matrix multiplication. SYM coupling is obtained by

[2] A single $5d$ SYM theory already contains the complete KK modes of the $6d$ theory [10, 11]. The KK modes are realized as the field configurations in R^4 carrying the nonzero Pontryagin number. Here, the θ-parameterized SYM field must have the zero Pontryagin number, because it is the zero mode of the $6d$ field along the $C_\mu(\theta)$ direction.

studying the scattering amplitude of the open strings ending on D branes. $[i,j]+[j,k]\to[i,k]$, so the $L^i_j M^j_k N^k_i$ type coupling is possible. On the other hand, for $M5$, we need to consider the scattering of the selfdual strings parallel to a given plane. Selfdual strings with the same orientation still have the SYM-coupling, while for selfdual strings with different orientations, the $[i,j]+[j,k]\to[i,k]$ relation is not valid, so the $[i,j]$ $(n/R_4, m/R_5)$ mode, the $[j,k]$ $(k/R_4, l/R_5)$ mode, and the $[k,i]$ $(-(n+k)/R_4, -(m+l)/R_5)$ mode of the $6d$ field cannot couple unless $nl=mk$, in which case all of them belong to the same $5d$ SYM theory with $\tan\theta=-\frac{nR_5}{mR_4}$. The difference between the SYM theory and the effective theory on $M5$'s is rooted in the fact that the boundary of the open string is the point, while the boundary of the open $M2$ is the line.

The $(n/R_4, m/R_5)$ momentum mode of the $6d$ SYM theory on $D5$ is dual to the open $F1$ ending on $D3$ with the winding number (n,m) around $x'_4\times x'_5$. From the scattering amplitude of the massive winding open strings and the massless open strings on $D3$, one can reconstruct the original $6d$ SYM theory. Similarly, for $M5$, we need to consider the interaction of the open (p,q) strings ending on $D3$ winding x'^{45} k times for all co-prime (p,q) and all nonnegative k, or in other words, the interaction for all of the monopoles and dyons in $\mathcal{N}=4$ SYM theory. When the scalar fields on $M5$ branes get the vacuum expectation value, $D3$ branes will be separated in the rest $5d$ transverse space, while the 3-string junctions [12, 13], which are also the bound states of the $[i,j]$ $[j,k]$ selfdual strings each carrying the transverse momentum in 45 plane, can be formed. The 3-string junction is characterized by three vectors (r_4, r_5), (s_4, s_5) and (t_4, t_5) in 45 plane, which may couple with the $(n/R_4, m/R_5)$ momentum modes as long as $(n,m)\propto(r_4,-r_5)$, or $(n,m)\propto(s_4,-s_5)$ or $(n,m)\propto(t_4,-t_5)$. The quantization of the 3-string junction with the lowest spin content gives the 1/4 BPS multiplet $V_4\otimes([1/2]\oplus[0]\oplus[0])$ with 2^6 states [14], which, when lifted to $6d$, becomes the $(2,1)$ multiplet with 2^7 states, among which, half are tri-fundamental and half are tri-anti-fundamental representation of $U(N)$. $[i,l]+[l,j,k]\to[i,j,k]$, $[j,m]+[i,m,k]\to[i,j,k]$, $[k,n]+[i,j,n]\to[i,j,k]$, so we may have couplings like $V^i_l T_{ijk}T^{ljk}$ or $V^i_l T_{ijk}T^{ljn}V^k_n$. [3]

However, with the given scalar vacuum expectation value $\vec{v}_i$ on $M5$, the bound states of the $[i,j]$ $\vec{v}_{ij}$ $(n/R_4, m/R_5)$ momentum mode and the $[j,k]$ $\vec{v}_{jk}$ $(k/R_4, l/R_5)$ momentum mode exist only under the certain condition $h(\vec{v}_{ij},\vec{v}_{jk}, n/R_4, m/R_5, k/R_4, l/R_5)>0$ with $h=0$ specifying the marginal stability curve [16]. Especially, when $\vec{v}_{ij}=0$, $\forall\, i,j$, except for $n=m=0$ or $k=l=0$, which is at the curve of the marginal stability, the rest bound states do not exist. The bound state of the $[i,j]$ $(0,0)$ momentum mode and the $[j,k]$ $(n/R_4, m/R_5)$ momentum mode could be taken as the tensionless selfdual string carrying the longitudinal momentum. It is unclear whether it is the bound state or just two separate states.

When compactified on x_5, the $[i,j]$ selfdual string extending along x_5 becomes the $[i,j]$ $F1$ localized in 1234 space. For the rest selfdual strings, to get the definite P_5 momentum,

[3] In [15], the scattering amplitude involving two charged $5d$ KK modes and one $5d$ zero mode for $M5$ branes compactified on S^1 is discussed. The charged KK mode is the 1/4 BPS dyonic instanton in $5d$ massive $(2,1)$ multiplet, while the zero mode is in $5d$ massless vector multiplet. The incorporation of the spin-3/2 particles of the $(2,1)$ multiplet into the theory requires a novel fermionic symmetry.

they must carry the definite transverse momentum thus are projected into the bound states of the $[i,j]$ $F1$ and the $[i,j]$ monopole string, each carrying the suitable P_4 and P_5 transverse momentum, localized in 123 space. The 1/2 BPS solutions for the BPS equations in $5d$ SYM theory match well with the above states, except for instantons, which is also localized in 1234 space. Similarly, the 3-string junctions in $6d$, when projected to $5d$, become the bound states of the $[i,j]$ $F1$ and the $[j,k]$ monopole string with P_4 and P_5 transverse momentum, localized in 123 space. Except for the dyonic instantons, the generic 1/4 BPS solutions in $5d$ SYM theory involving no more than three $D4$'s have a one-to-one correspondence with these states.

The rest of this paper is organised as follows: In section 2, we discuss the longitudinal momentum mode on branes with special emphasis on the point-like charged 1/2 BPS instantons on $D4$ branes. In section 3, we consider the selfdual strings on $M5$ branes parallel to the 45 plane, or equivalently, the KK momentum mode of the $6d$ theory upon the compactification on $x_4 \times x_5$. In section 4, we study the interaction of the $6d$ $(2,0)$ theory by considering its KK mode on $x_4 \times x_5$. In section 5, we consider various momentum-carrying BPS states in $5d$ SYM theory, especially, the monopole string carrying the longitudinal momentum and the string carrying $D0$ charge. In section 6, we discuss the states living at the triple intersection of $M5$ branes. The discussion is in section 7.

2 The point-like charged P_5 momentum mode on $D4$ and the nonabelian $5d$ massive $(2,0)$ tensor multiplet

In this section, we discuss the longitudinal momentum mode on $M5$ branes, which is carried by selfdual strings. We will argue that there is no closed charged selfdual strings and so no classical point-like charged momentum mode on $M5$ branes. Nevertheless, on N coincident $D4$ branes, the localized $[i,j]$ P_5 mode (D0 brane) could be realized as the quantum superposition of the $[i,j]$ P_5 mode in momentum eigenstate of, for example, P_4, which is the $[i,j]$ tensionless monopole string wrapping x_4, carrying the $D0$ charge as well as the P_4 longitudinal momentum, or equivalently, the $[i,j]$ $(n,1)$ string on $D3$ if x_4 is compactified. The $5d$ massive nonabelian $(2,0)$ tensor multiplet with mass $P_5 = 1/R_5$ could then be decomposed into a tower of $4d$ massive $U(N)$ vector multiplet arising from the quantization of the $(n,1)$ open strings for all $n \in \mathbf{Z}$.

2.1 The longitudinal momentum mode on branes

Let us first see the transverse momentum of the branes. For a Dp brane with the transverse dimension x_{p+1} compactified to a circle S^1, the brane may locate at a particular point in S^1 or have the definite momentum along S^1. In the former situation, after the T-duality transformation along x_{p+1}, Dp becomes $D(p+1)$ with the gauge field A_{p+1} getting the vacuum expectation value. In the latter case, the T-duality transformation converts Dp

into $D(p+1)$ with the definite electric flux $F_{0(p+1)}$. $D(p+1)$ cannot have the definite A_{p+1} and $F_{0(p+1)}$ simultaneously, just as Dp cannot have the definite X_{p+1} and P_{p+1} at the same time. In M theory, if x_5 is compactified to S^1, $M2$ transverse to x_5 may either locate at a particular point in S^1 or have the definite P_5 momentum. $M2$ with zero P_5 momentum is $D2$ in type IIA string theory. $M2$ with nonzero P_5 momentum is the $D2$-$D0$ bound state. The transverse velocity should be the same everywhere on $D2$, so the $D0$ charges are uniformly distributed over $D2$. If the masses of the $D2$ and $D0$ are m_2 and m_0 respectively, the energy of the $D2$-$D0$ bound state is $\sqrt{m_2^2+m_0^2}$, in contrast to the energy of the $D4$-$D0$ bound state, which is m_4+m_0.

As for the longitudinal momentum mode on branes, consider Dp brane with the longitudinal dimension x_p compactified to a circle S^1, Dp may carry momentum along x_p. Under the T-duality transformation in p direction, we get the D_{p-1}-$F1$ bound state with $F1$ ending on D_{p-1} winding the transverse circle x'^p. If the i_{th} and the j_{th} D_{p-1} branes are separated along another transverse dimension x'^{p+1}, the closed $F1$ becomes open, which, under the T-duality transformation along x'^p, gives the open $[i,j]$ string ending on Dp branes carrying the P_p momentum. So, more precisely, the P_p longitudinal momentum of the Dp brane is the P_p transverse momentum of the open strings living on them. Consider the i_{th} and the j_{th} Dp branes with the $[i,j]$ string orthogonally connecting them carrying momentum P_p, the total energy is $m_p+\sqrt{T_{F1}^2|\vec{v}_i-\vec{v}_j|^2+P_p^2}$, where m_p and $T_{F1}|\vec{v}_i-\vec{v}_j|$ are masses of Dp and $[i,j]$ string respectively. When $\vec{v}_i=\vec{v}_j$, the energy reduces to $m_p+|P_p|$, corresponding to the Dp branes carrying the $[i,j]$ P_p longitudinal momentum. The low energy effective action on N coincident Dp branes is the $U(N)$ SYM theory. When compactified on x_p, one may get an infinite tower of KK modes still in the adjoint representation of $U(N)$. We have seen that the P_p momentum carries charge, thus is indeed in the adjoint representation.

For $M2$ branes, two $M2$ branes orthogonally intersecting at a point may form the threshold bound state, so the transverse momentum of one $M2$ gives the longitudinal momentum of the other, as long as the two can keep intersecting at one point. Similarly, it is natural to expect that for $M5$ branes, the P_k longitudinal momentum may actually be the P_k transverse momentum of the selfdual strings living in them. Especially, for $5d$ SYM theory, the P_5 momentum may just come from the monopole strings living in $D4$. However, P_5 like this is distributed in a straight line other than localized at a point. Localized instantons do exist, which are $D0$ branes resolved in $D4$. The line-like P_5 momentum carried by selfdual strings has the $[i,j]$ charge, while the point-like P_5 momentum corresponding to $D0$ branes is chargeless. It is necessary to construct the point-like 1/2 BPS momentum mode carrying charge.

One may want to consider the closed selfdual strings, which, with the length shrinking to zero, may carry the point-like momentum. However, the selfdual strings carrying charge must extend along a straight line. We cannot get the closed charged selfdual strings unless the worldvolume of the $M5$ branes has the nontrivial 1-cycle. Consider the $[i,j]$ selfdual string segment extending along ABC, where A B C are three points in $M5$. Suppose

$AB \perp BC$, then the AB, BC strings are actually the same as the $[i,j]$ $F1$ and the $[i,j]$ $D1$. The configuration like this is not BPS, so $F1$ and $D1$ may exert force to each other. The $[i,j]$ $F1$-$D1$ bound state is not at the threshold and has the mass $|\vec{v}_{ij}|\sqrt{|AB|^2 + |BC|^2}$ due to the binding energy. We are actually talking about the $[i,j]$ selfdual string segment extending along AC. The $[i,j]$ selfdual string cannot vibrate freely, because different parts may exert force to each other.

On the other hand, the chargeless selfdual strings do not have this problem. They can be closed and may carry the point-like momentum. One such example is the $[i,i]$ selfdual string. On the i_{th} $M5$ brane, we have the zero length $[i,i]$ closed selfdual string, or in other words, the collapsed $M2$ brane, the quantization of which gives the expected $U(1)$ $(2,0)$ tensor multiplet [3, 4, 5]. When compactified on x_5, the point-like $D2$ with P_5 momentum becomes the $D0$ confined to the i_{th} $D4$. Another example is the little string theory [6, 17, 18]. Consider N coincident type IIA $NS5$ branes with the longitudinal dimension x_5 compactified to a circle with the radius R_5. The 11_{th} dimension is x_{10} which is compactified with the radius R_{10}. There are closed fundamental strings with tension $2\pi R_{10} T_{M2}$ living in $NS5$, which are closed $M2$'s wrapping x_{10} intersecting $M5$ along a closed curve. After a series of duality transformations, the momentum mode (carried by the type IIA string) along x_5 is converted to the $D0$ branes living in $D4$ branes with the compactified transverse dimension x'_5. If the original type IIA closed string has the finite length, we will get a closed $D2$ wrapping x'_5 intersecting $D4$ along a closed curve carrying the uniformly distributed $D0$ charge. The $D0$ branes are obtained when the size of the $D2$ brane carrying them shrinks to zero. Actually, the type IIA $NS5$ brane picture and the $D4$ brane picture are S-dual to each other with 5 and 10 switched. On type IIA $NS5$ branes, the P_5 momentum is carried by the closed string. When the string shrinks to a point, we simply take it as a momentum mode without the string involved. Similarly, on $D4$ branes, we may have closed $D2$ carrying P_5 momentum. With the closed $D2$ shrinking to a point, we are left with the $D0$ brane/P_5 momentum.

On a single $NS5$ brane, purely P_5 momentum is carried by strings that do not wind x_5, or alternatively, $M2$'s that do not wrap x_5. Correspondingly, on a single $D4$ brane, the purely P_5 momentum mode should be carried by the closed $D2$ branes other than strings. The complete KK modes on $NS5$ branes upon the compactification on x_5 are characterized by (m,n), where m and n are the winding number and the momentum mode of the string along x_5 respectively. The (m,n) mode has the mass $2\pi m T_{F1} R_5 + n/R_5$. $(m,0)$ mode, $(0,n)$ mode and (m,n) mode are in the $5d$ $(1,1)$, $(2,0)$ and $(2,1)$ multiplets preserving 1/2, 1/2, and 1/4 supersymmetries respectively [19]. For the dyonic strings in [11], with x_6 compactified to a circle, the bound state of the $[1,2]$ and $[2,1]$ dyonic strings carrying n instanton number is just be the $(1,n)$ mode here. The generic (m,n) mode is obtained by the quantization of the type IIA strings. Especially, with $N_L = 0$, the level-matching condition requires $N_R = mn$, so the oscillation mode along the string must be turned on [19, 20]. This is easy to understand. For type IIA string wrapping x_5, P_5 can only come from the internal oscillation since there is no transverse momentum along x_5. On the other hand, if the $[i,j]$

selfdual strings wrapping x_5 cannot oscillate, the P_5 momentum carried by it can only come from selfdual strings extending in 1234 space. The (m,n) mode in this case is actually the threshold bound state of the $(m,0)$ mode and the $(0,n)$ mode. The former is associated with the selfdual string extending along x_5, while the latter is given by the tensionless selfdual strings extending along 1234 space carrying the P_5 momentum. It is possible for the $[i,j]$ $(m,0)$ mode and the $[j,k]$ $(0,n)$ mode to form the threshold bound state with ijk indices, which we will discuss later.

2.2 The point-like charged P_5 momentum mode

Now, let us consider the relation between the point-like charged P_5 momentum mode and the line-like charged P_5 momentum mode. For N coincident $D4$ branes with the longitudinal dimension x_4 compactified, the T-duality transformation along x_4 converts the $D4$-$D0$ bound state into the $D3$-$D1$ bound state with $D1$ winding x'^4. More precisely, $D1$ carrying the definite electric flux, or equivalently, $D1$-$F1$ bound state, corresponds to $D0$ in P_4 momentum eigenstate, while $D1$ with the definite A_4 field corresponds to $D0$ in X_4 position eigenstate. We may take the $(n,1)$ strings in $D3$ as the bases, the superposition of which gives $D1$ with the definite A_4, which is also the $D0$ located at a definite point in $D4$. Moreover, since $D1$ ending on $D3$'s can also carry charge, the $[i,j]$ $D1$ wrapping x'^4 is dual to the $[i,j]$ $D2$ wrapping x_4 with the $D0$ charge spreading over $\vec{v}_{ij} \times x_4$. When $\vec{v}_{ij}=0$, we are left with the tensionless $[i,j]$ monopole string winding x_4 carrying the uniformly distributed $D0$ charge, which could also be taken as the $[i,j]$ $D0$ in P_4 momentum eigenstate with the eigenvalue 0. Similarly, the $[i,j]$ $(n,1)$ string is dual to the combination of the tensionless $[i,j]$ monopole string winding x_4 carrying the uniformly distributed $D0$ charge and the massless $[i,j]$ string carrying P_4 momentum, which could be simply taken as the $[i,j]$ $D0$ with the nonzero P_4 eigenvalue. Still, the superposition of the $[i,j]$ $(n,1)$ strings for all n gives the $[i,j]$ $D0$ in X_4 position eigenstate. The instanton solutions describing the $[i,j]$ $D0$ with the definite P_4 momentum have the translation invariance along x_4, involving both magnetic and the electric fields. These states compose the complete spectrum for the charged $D0$ living in $D4$, while the localized charged $D0$ is the superposition of them.

The above conclusion can be stated in the language of $M5$ branes, since the bound state of the tensionless $[i,j]$ monopole string wrapping x_4 carrying $D0$ charge and the massless $[i,j]$ string carrying the P_4 momentum is just the tensionless selfdual string living in $x_4 \times x_5$ carrying the transverse (P_4, P_5) momentum. Consider $M5$ branes with x_4 and x_5 compactified with the radii R_4 and R_5, $B_{45} = \frac{1}{2\pi R_4 R_5}$. Tensionless selfdual string winding x_4 and x_5 m and n times may carry the transverse momentum $(-n/R_4, m/R_5)$, thus could be described by the wave function

$$\frac{1}{4\pi^2 R_4 R_5} \exp\left\{ i(-\frac{nx_4}{R_4} + \frac{mx_5}{R_5}) \right\} \delta(x_1 - X_1)\delta(x_2 - X_2)\delta(x_3 - X_3). \tag{1}$$

The P_5 momentum localized in x_4 has the wave function

$$\frac{1}{2\pi R_5}\exp\left\{\frac{imx_5}{R_5}\right\}\delta(x_1-X_1)\delta(x_2-X_2)\delta(x_3-X_3)\delta(x_4-X_4), \tag{2}$$

which is the superposition of (1) with all $n\in\mathbf{Z}$. In this respect, at least for $M5$ with at least two dimensions compactified, the longitudinal momentum is still given by the basic excitations, which are the selfdual strings here. The only difference is that the selfdual string is the one dimensional object, so the transverse momentum carried by it will appear as the two dimensional wave other than the one dimensional wave like the momentum carried by particles. To get the complete spectrum, we need the particles with the location covering x_4, or the selfdual strings with the orientation covering the 45 plane. $\{\delta(x_4-X_4)|X_4\in[0,2\pi R_4)\}$ and $\left\{\frac{1}{2\pi R_4}e^{\frac{inx_4}{R_4}}|n\in\mathbf{Z}\right\}$ are different bases for the same Hilbert space.

On $D4$ branes, the $[i,j]$ P_5 momentum can only be carried by the $[i,j]$ selfdual strings. There is no classical solution for the point-like instanton with the charge $[i,j]$. However, we do have the solution for the chargeless point-like instantons, which may consist of N instanton partons with charge $[1,2]$, $\cdots$, $[N-1,N]$, $[N,1]$, while the size ρ is the parameter characterizing the distance between the instanton partons [21, 22]. Similarly, for type IIA $NS5$ branes with x_5 compactified, the P_5 momentum is carried by the point-like closed strings, which are also composed by the $[1,2]$, $\cdots$, $[N-1,N]$, $[N,1]$ closed selfdual strings from M theory's point of view. It is difficult to get a single closed selfdual string with charge $[i,j]$. However, if one longitudinal dimension of $D4$ is compactified, an instanton on $D4$ branes will be dual to a D-string on $D3$ branes winding the transverse circle one time. A closed D-string is composed by the $[1,2]$, $\cdots$, $[N-1,N]$, $[N,1]$ D-string segments [21, 22]. The $[i,j]$ D-string segment can exist independently, because it is the $[i,j]$ monopole string extending along the compactified longitudinal dimension carrying the transverse P_5 momentum. Similarly, for the P_4 longitudinal momentum mode on $D4$ branes, we have the $[i,j]$ P_4 mode carried by the $[i,j]$ open string, which is the $[i,j]$ selfdual string winding x_5 one time, carrying the transverse P_4 momentum thus could also exist separately. The $[1,2]$, $\cdots$, $[N-1,N]$, $[N,1]$ P_4 modes can combine together to give a chargeless P_4 mode as well, but it is not necessary anymore.

2.3 The $5d$ $U(N)$ massive $(2,0)$ tensor multiplet

For N coincident $M5$ branes with the compactified x_5, the $6d$ $(2,0)$ tensor multiplet could be decomposed into the zero mode and the KK modes. The zero mode is in $5d$ $U(N)$ vector multiplet, while the KK modes are in $5d$ massive $(2,0)$ tensor multiplets. The point-like $[i,j]$ P_5 mode (the $[i,j]$ $D0$ brane) is the the superposition of the tensionless $[i,j]$ selfdual strings extending in, for example, the 45 plane, carrying the transverse momentum (P_4,P_5) with the same P_5 but all P_4. Corresponding, the $5d$ $(2,0)$ tensor multiplet is then decomposed into the sum of the $4d$ KK modes in $U(N)$ vector multiplet, arising from the quantization of the above selfdual string states.

Consider $B_{\mu\nu}$ in $5d$ $U(1)$ massive $(2,0)$ multiplet with mass $1/R_5$. $B_{\mu\nu}$ satisfies the selfduality condition

$$B_{\mu\nu} = -\frac{iR_5}{2}\epsilon_{\mu\nu\lambda\rho\sigma}\partial^\lambda B^{\rho\sigma} \tag{3}$$

and the equation of motion

$$\partial^\lambda\partial_\lambda B_{\mu\nu} + \frac{1}{R_5^2}B_{\mu\nu} = 0, \tag{4}$$

where $\mu,\nu,\lambda,\rho,\sigma = 0,1,2,3,4$ [11]. Do a further compactification on x_4,

$$B_{\mu\nu} = \sum_k e^{ikx_4/R_4}B_{\mu\nu}^{(k)}. \tag{5}$$

Due to (3), for $i,j = 0,1,2,3$ and $k \in Z$, $B_{ij}^{(k)}$ could be expressed in terms of $B_{i4}^{(k)}$ thus could be dropped. We are left with a tower of the $4d$ massive vector field $A_i^{(k)} = B_{i4}^{(k)}$ satisfying the constraint

$$\partial^i A_i^{(k)} = 0 \tag{6}$$

as well as the equation of motion

$$\partial_j\partial^j A_i^{(k)} + (\frac{1}{R_5^2} + \frac{k^2}{R_4^2})A_i^{(k)} = 0. \tag{7}$$

Each $A_i^{(k)}$ carries 3 degrees of freedom, the same as $B_{ij}^{(k)}$. In 45 plane, the $(n,1)$ string carries the momentum $(n/R_4, 1/R_5)$ thus gives the $4d$ vector multiplet $A_i^{(n)}$. All of the $A_i^{(n)}$ are on the equal footing, which is consistent with the S-duality. To account for the P_5 momentum m/R_5 with $m > 1$, we need (n,m) strings, so altogether, all (n,m) strings should be included to give the complete $6d$ dynamics. Under the compactification on x_4 and x_5, the $6d$ field $B_{\alpha\beta}$ with $\alpha,\beta = 0,\cdots,5$ is decomposed into the $4d$ KK modes $(n/R_4, m/R_5)$ corresponding to the (n,m) string. Each KK mode gives a $4d$ massive vector field $A_i^{(n,m)}$, for which, the constraint and the equation of motion could be obtained by replacing R_5 and k in (7) by R_5/m and n.

Extending the discussion to the nonabelian case is a little difficult, since we don't know the equations for the nonabelian tensor field. However, we do know that the $(n/R_4, 0)$ mode, which is the KK mode of the $5d$ massless SYM field, is in $4d$ adjoint massive vector multiplet. The rest $(n/R_4, m/R_5)$ modes are related with $(n/R_4, 0)$ via the S-duality, so they should also form the $4d$ adjoint massive vector multiplet. The whole KK tower of the $4d$ vector multiplet together may give the $6d$ $(2,0)$ tensor multiplet in adjoint representation of $U(N)$.

3 Selfdual strings with the orientation covering a plane and the $M5$-$D3$ duality

In this section, we will directly discuss the $M5$ branes and show that selfdual strings parallel to a given plane could offer the complete degrees of freedom on $M5$. The proposal is also

supported by the $M5$-$D3$ duality, in which, the KK mode of $M5$ on $x_4 \times x_5$ is dual to the (p,q) open strings ending on $D3$ winding the transverse x'^{45}. When $M5$ branes are separated in $5d$ transverse space, on $D3$, 3-string junctions may form, which, in $M5$ picture, is the bound state of the unparallel selfdual strings.

3.1 Selfdual strings parallel to the 45 plane and the momentum mode of $M5$ on $x_4 \times x_5$

In previous discussion, we have seen that selfdual strings extending along all possible directions in 45 plane may give the complete spectrum for a single $6d$ particle. The common eigenstates of $\hat{X}_1$ $\hat{X}_2$ $\hat{X}_3$ $\hat{P}_4$ $\hat{P}_5$,

$$\Lambda = \left\{ \delta(x_1 - X_1)\delta(x_2 - X_2)\delta(x_3 - X_3)e^{iP_4 x_4}e^{iP_5 x_5} | \forall\, X_1, X_2, X_3, P_4, P_5 \right\} \tag{8}$$

may be the suitable bases, the superposition of which can give a $6d$ particle localized in $(X_1, X_2, X_3, X_4, X_5)$. Although the position eigenstates can also be obtained, the basic excitations are $[i,j]$ selfdual strings other than the $[i,j]$ particles. Since it is (8) other than the position eigenstates that is naturally realized, the KK modes in this theory may tell us more than the KK modes in theories with point-like excitations.

Until now, our discussion is only restricted to coincident $M5$ branes. When $\vec{v}_{ij} \neq 0$, the selfdual string carrying (P_4, P_5) momentum are massive. Consider the $[i,j]$ selfdual strings with the length and the orientation characterized by the vector (qR_4, pR_5) in 45 plane. p and q are co-prime, so the selfdual string only winds $x_4 \times x_5$ once. In 123 space, the string is localized at a point. The Wilson surface in $x_4 \times x_5$ is trivial. Nevertheless, each $[i,j]$ string can still effectively pick up the background 2-form field $B_{45} = k/(2\pi R_4 R_5)$, $\forall\, k \in \mathbf{N}$. $\forall\, m, n \in \mathbf{Z}$, $\exists\, k, p, q$, $m = kq$, $n = kp$. In 45 plane, the $[i,j]$ string will get the definite transverse momentum $(n/R_4, -m/R_5)$, thus could be taken as the plane wave $e^{i(\frac{nx_4}{R_4} - \frac{mx_5}{R_5})}$. If the momentum in 123 space is (P_1, P_2, P_3), the energy will be

$$E = \sqrt{P_1^2 + P_2^2 + P_3^2 + \frac{n^2}{R_4^2} + \frac{m^2}{R_5^2} + 4\pi^2 |\vec{v}_{ij}|^2 (q^2 R_4^2 + p^2 R_5^2)}, \tag{9}$$

since the $[i,j]$ selfdual string has the rest mass $2\pi |\vec{v}_{ij}| \sqrt{q^2 R_4^2 + p^2 R_5^2}$.

Notice that there is an ambiguity for the mass of the zero mode in $4d$. With $k = 0$, any (qR_4, pR_5) string can be the $4d$ zero mode with mass $2\pi |\vec{v}_{ij}| \sqrt{q^2 R_4^2 + p^2 R_5^2}$. However, the zero mode is unique. In the dual $D3$ picture, there are unwrapped $[i,j]$ (p,q) strings with the length $|\vec{v}_{ij}|$. One may choose one possible $(qR_4, pR_5)/(p,q)$ as the zero mode. A particular S-frame is selected in this way, while in other S-frames, all (qR_4, pR_5) can get the chance to act as the zero mode. For the point-like excitations, the $5d$ momentum can uniquely fix the state; however, for the line-like excitations, with the given $5d$ momentum, the selfdual string orientations can still vary in a $4d$ space orthogonal to the momentum. If the selfdual string orientations are restricted to a plane, a one-to-one correspondence between the momentum

and state may be realized except for the momentums orthogonal to that plane. So, the fixing of the S-frame is necessary. As is shown in later discussions, the $M5$-$D3$ duality also intrinsically involves the selection of the S-frame.

For $M5$ compactified on x_5, the P_5 zero mode should be the state with the zero momentum along x_5, localized in 1234 space. Selfdual string extending along x_5 is the only one meeting the requirement, and so, in Coulomb branch, the mass of the W-bosons is $2\pi|\vec{v}_{ij}|R_5$ without the ambiguity. Notice that there is a distinction between the little string theory and the theory on $M5$ branes. For little string theory with x_5 compactified, there are momentum mode and the winding mode. The momentum mode is carried by the closed string. Although the string has the finite tension, the mass of the momentum mode is still m/R_5, because the string without winding x_5 can shrink to point thus has the zero mass and has no contribution to the energy. On the other hand, for $6d$ $(2,0)$ theory with x_5 compactified, the zero mode is the $5d$ SYM field. In Coulomb branch, the mass of the $[i,j]$ zero mode is $2\pi|\vec{v}_{ij}|R_5$ other than 0, since the zero mode is actually the selfdual string winding x_5 once. There is no way to get rid of the lowest winding mode, because we do not have the closed charged selfdual string, while the straight selfdual string localized in 1234 space must extend along x_5.

The direct study of the $6d$ $(2,0)$ theory is difficult. The x_5 compactification will give the $5d$ massive tensor multiplet, which is also not quite accessible. The $4d$ KK modes upon the compactification on $x_4 \times x_5$ are relatively easy to study. Moreover, the previous discussion indicates that (8) might be the suitable bases to consider the $6d$ theory, so in the following, we will focus on the $4d$ KK modes arising from the $6d$ theory.

3.2 The $M5$-$D3$ duality

Actually, $M5$ with x_4 and x_5 compactified in $11d$ spacetime is dual to $D3$ with one transverse dimension x'^{45} compactified in $10d$ spacetime. Just as $M5$ with x_5 compactified in M theory is dual to $D4$ in type IIA string theory with x_5 being the M theory circle, $M5$ with x_4 and x_5 compactified in M theory is dual to $D3$ in type IIB string theory with x'^{45} compactified. If the radii of x_4 and x_5 are R_4 and R_5, the radius of x'^{45} will be $R'_{45} = 1/(2\pi R_4 T_{F1}) = 1/(2\pi R_5 T_{D1})$. The five transverse dimensions of $M5$, x^I, are dual to the rest five transverse dimensions of $D3$, x'^I. $I = 6 \cdots 10$. If the scalar fields on $M5$ branes get the vacuum expectation value Φ_i^I with $i = 1 \cdots N$, $D3$ branes will be separated along x'^I with the transverse positions

$$X_i'^I = 2\pi R_5 \Phi_i^I / T_{F1} = 2\pi R_4 \Phi_i^I / T_{D1}. \tag{10}$$

$x_4 \times x_5$ is dual to x'^{45}. If the B_{45} on $M5$ branes gets the vacuum expectation value B_{45i}, $D3$ branes will be separated along x'^{45} with the transverse positions

$$X_i'^{45} = 2\pi R_5 B_{45i} / T_{F1} = 2\pi R_4 B_{45i} / T_{D1}. \tag{11}$$

If the $B_{\mu 5}$ for $\mu = 1, 2, 3$ on $M5$ branes gets the vacuum expectation value $B_{\mu 5i}$, the gauge field $A_{\mu i}$ on $D3$ branes will get the vacuum expectation value

$$A_{\mu i} = 2\pi R_5 B_{\mu 5i}. \tag{12}$$

(12) indicates that the $(0,0)$ mode of $M5$ is dual to the $(1,0)$ string on $D3$ with the winding number 0. A particular S-frame is selected.

For T-duality, Dp with the longitudinal x_p compactified is dual to D_{p-1} with the transverse x'^p compactified, with A_p converted to X^p, the momentum mode along x_p transformed to the winding mode along x'^p. For $M5$, we have $B_{\mu\nu}$ instead of A_μ, so two longitudinal dimensions x_4 x_5 are transformed to one transverse dimension x'^{45}, while the (P_4, P_5) momentum modes become the winding modes of the (p,q) strings along x'^{45} for all co-prime p q.

The $M5$-$D3$ duality requires that the both sides have the same degrees of freedom. Especially, the three vector fields and one scalar field on $D3$ are dual to the four 2-form fields $B_{\mu 5}$ on $M5$. The rest 2-form fields on $M5$ have no counterpart thus could be neglected. This is consistent with the self-duality condition on $M5$. Especially, for $M5$ compactified on $x_4 \times x_5$,

$$B_{\mu\nu}(x_4, x_5, \vec{x}) = \frac{1}{2\pi\sqrt{R_4 R_5}} \sum_{n,m} e^{i(nx_4/R_4 + mx_5/R_5)} B_{\mu\nu}^{(n,m)}(\vec{x}). \tag{13}$$

The zero mode has no winding number around x'^{45}. $B_{45}^{(0,0)}(\vec{x}) \to X^{45(0,0)}(\vec{x})$, $B_{i5}^{(0,0)}(\vec{x}) \to A_i^{(0,0)}(\vec{x})$, where $i = 1,2,3$. The rest $B_{\mu\nu}^{(0,0)}$ could be neglected. The bosonic degrees of freedom are $6+2=8$. The higher mode has the nonzero x'^{45} winding number, so there is no $X^{45(n,m)}$ for $m,n \neq 0$. $B_{i5}^{(n,m)}(\vec{x}) \to A_i^{(n,m)}(\vec{x})$, where $i = 1,2,3$. $B_{45}^{(n,m)}(\vec{x})$ together with $A_i^{(n,m)}(\vec{x})$ gives $4-1=3$ gauge degrees of freedom, so the total bosonic degrees of freedom are still $5+3=8$.

The $[i,j]$ (qR_4, pR_5) selfdual string on $M5$ is dual to the $[i,j]$ (p,q) string on $D3$. If x_4 and x_5 are compact, x'_{45} will also be compact, so even if $B_{45}=0$, the covering space of x'_{45} will still have N coincident $D3$ branes distributed with the period $2\pi R'_{45}$. $\forall\, i,j$, we have the $[i,j]$ (p,q) string connecting the i_{th} and the j_{th} $D3$ branes with the length $2\pi k R'_{45}$, corresponding to the $[i,j]$ (qR_4, pR_5) selfdual string coupling with the 2-form field $B_{45} = k/(2\pi R_4 R_5)$, getting the momentum $(kp/R_4, -kq/R_5)$. If the other transverse fields on $D3$ also get the vacuum expectation value, the mass of the $[i,j]$ (p,q) string will be

$$M = 2\pi\sqrt{q^2R_4^2 + p^2R_5^2}\sqrt{(\frac{k}{2\pi R_4 R_5})^2 + |\vec{\Phi}_{ij}|^2}, \tag{14}$$

which is the same as the energy of the $[i,j]$ (qR_4, pR_5) selfdual string.

The $(kp/R_4, -kq/R_5)$ momentum mode is dual to the $[i,j]$ (p,q) string winding x'^{45} k times. The (p,q) string with all possible winding numbers gives a $5d$ SYM theory whose basic excitations are (p,q) strings. In this way, the $4d$ KK modes are equivalent to a series of $5d$ SYM fields labeled by (p,q) with p and q co-prime. The (p,q) $5d$ SYM fields, when lifted to $6d$, are translation invariant along the (qR_4, pR_5) direction. They are the fields related with the (qR_4, pR_5) selfdual strings. With all (p,q) included, the selfdual string orientation then covers the whole 45 space.

3.3 The 3-string junctions on $D3$ and $M5$

When $\vec{v}_i = 0$, all $D3$ branes are separated along a straight line, so the possible BPS states are still the original 1/2 BPS states. To get the new states, the vacuum expectation values of the five scalar fields on $M5$ branes must be turned on. In $D3$ brane picture, $D3$'s then appear as N arbitrary points in the $5d$ transverse space orthogonal to x'^{45}. The only possible new BPS states are 1/4 BPS 3-string junctions, which are also the bound states of the $[i,j]$ (p,q) string and the $[j,k]$ (r,s) string. On $M5$ side, the 3-string junction is the bound state of the $[i,j]$ $[j,k]$ selfdual strings each carrying the transverse momentum $(kp/R_4, -kq/R_5)$ and $(hr/R_4, -hs/R_5)$[4].

We are interested with the N coincident $M5$ branes, since in that case, the states can be massless in six dimensional sense thus will contribute to the entropy. We have seen that the bound states of the KK mode cannot give the new degrees of freedom, but we haven't considered the bound states of the KK mode and the zero mode, which, for example, can be taken as the tensionless $[i,j]$ $(0, R_5)$ string. In $D3$ brane picture, that is the massless $[i,j]$ $(1,0)$ string, which may form the 1/4 BPS threshold bound state with any $[j,k]$ (p,q) strings with the length $2\pi k R'_{45}$. In $6d$, they are the $[i,j]$ $(0,R_5)$ and the $[j,k]$ (qR_4, pR_5) tensionless selfdual strings located at the same point in 123 space. The former has the zero momentum in 45 plane, while the later carries the transverse momentum $(kp/R_4, -kq/R_5)$. A potential problem is that the threshold bound state may decay. If they do decay, then there will be no three indexed BPS states on $M5$.

4 The interaction of the $6d$ $(2,0)$ theory seen from its KK modes on $x_4 \times x_5$

We now turn to the $6d$ $(2,0)$ theory on $M5$ branes. The 3-algebra valued $(2,0)$ tensor multiplet with the constant vector C_μ proposed in [9] is the natural framework to describe the selfdual strings. Selfdual strings parallel to the 45 plane give the fields $f_{(\theta)}(x_\mu)$ with $C_\mu(\theta)\partial^\mu f_{(\theta)}(x_\mu) = 0$ for $C_\mu(\theta) = \cos\theta\delta_\mu^4 + \sin\theta\delta_\mu^5$, which altogether are equivalent to the $6d$ field. $f_{(\theta)}(x_\mu)$ is in the adjoint representation of $U(N)$. Couplings like $Tr[f_{(\theta_1)}\cdots f_{(\theta_n)}]$ do not exist unless $\theta_1 = \cdots = \theta_n$, because the bound state of two unparallel selfdual strings is the string junction other than another selfdual string. The quantization of the 3-string junction gives the $(2,1)$ multiplet $g_{(\theta_1,\theta_2,\theta_3)}(x_\mu)$ in $N \times N \times N$ and $\bar{N} \times \bar{N} \times \bar{N}$ representation of $U(N)$, which may couple with the vector multiplet $f_{(\theta)}(x_\mu)$ as long as $\theta = \theta_1, \theta_2, \theta_3$. On coincident $M5$ branes, $g_{(\theta_1,\theta_2,\theta_3)}(x_\mu)$ reduces to $g_{(\theta)}(x_\mu)$ subject to the constraint $C_\mu(\theta)\partial^\mu g_{(\theta)}(x_\mu) = 0$, giving a $6d$ field.

[4]See [23] for another discussion of the string junctions on $M5$ and $D4$.

4.1 $6d$ field decomposed into the θ-parameterized $5d$ fields

Recall that in [9], the equations of motion for the 3-algebra valued $(2,0)$ tensor multiplet involve a constant vector field C_μ, where $\mu = 0 \cdots 5$, giving a direction along which all of the fields are required to be translation invariant. The theory with the fixed C_μ describes the selfdual string extending along it. The selfdual string has the zero momentum along C_μ but may get the arbitrary momentum along the four transverse dimensions, so the theory describing it is just the $5d$ $U(N)$ SYM theory, which is the reduction of the $6d$ $(2,0)$ theory along C_μ. Moreover, as the zero mode along C_μ, the field configurations of the $5d$ SYM theory on R^4 should carry the zero instanton number (Pontryagin number) thus are topologically trivial on the equivalent S^4. To recover the full $6d$ theory, we need the selfdual strings with the orientations covering all directions in a plane, which, for definiteness, is taken as the 45 plane. Correspondingly, C_μ is replaced by $C_\mu(\theta) = \cos\theta\delta^4_\mu + \sin\theta\delta^5_\mu$, while the original fields $f(x_\mu)$ now become $f(\theta, x_\mu)$ still with the constraint $C_\mu(\theta)\partial^\mu f(\theta, x_\mu) = 0$, giving rise to a $6d$ field.

Suppose the $U(N)$ $6d$ $(2,0)$ tensor multiplet field configuration is given. For simplicity, consider the scalar fields $X^I(x_m, x_4, x_5)$, where $m = 0,1,2,3$, $I = 6,7,8,9,10$.

$$\int dx_\theta\, X^I(x_m, \cos\theta x_\theta + \sin\theta y_\theta, -\sin\theta x_\theta + \cos\theta y_\theta) = \Phi^I(x_m, \theta, y_\theta) \tag{15}$$

is the scalar field in the $5d$ SYM theory related with θ. X^I and Φ^I have the scaling dimensions 2 and 1 respectively. Φ^I is the zero mode of X^I along $C_\mu(\theta)$. Also, notice that

$$\int dx_\theta dy_\theta\, X^I(x_m, \cos\theta x_\theta + \sin\theta y_\theta, -\sin\theta x_\theta + \cos\theta y_\theta) = \phi^I(x_m) \tag{16}$$

is independent of θ. ϕ^I is the zero mode of $\Phi^I(x_m, \theta, y_\theta)$ in the $4d$ spacetime. All of the $5d$ SYM theories share the same zero mode in $4d$, because the $6d$ theory has the unique zero mode in $4d$. The vector field A and the spinor field η with the scaling dimensions 1 and $3/2$ in $5d$ SYM theory could be constructed in the similar way from the $6d$ 2-form field B and the spinor field Ψ with the scaling dimensions 2 and $5/2$. Since the integration is carried out along a particular direction, more precisely, the original scalar fields, 2-form field, and the spinor field are converted into the vector fields, vector field, and the spinor-vector field respectively.

One may also want to reconstruct X^I from Φ^I.

$$X^I(x_m, x_4, x_5) = \int dp_4 dp_5\, e^{i(p_4x_4 + p_5x_5)}\phi^I_{(p_4,p_5)}(x_m). \tag{17}$$

If x_4 and x_5 are compact, $p_4 = k\bar{p}_4/R_4$, $p_5 = -k\bar{p}_5/R_5$, $(\bar{p}_4, \bar{p}_5)$ is the co-prime pair,

$$X^I(x_m, x_4, x_5) = \sum_{(\bar{p}_4,\bar{p}_5)}\sum_k e^{ik(\frac{\bar{p}_4x_4}{R_4} - \frac{\bar{p}_5x_5}{R_5})}\phi^I_{(\bar{p}_4,\bar{p}_5;k)}(x_m) = \sum_{(\bar{p}_4,\bar{p}_5)}\Phi^I(x_m, \bar{p}_4R_5x_4 - \bar{p}_5R_4x_5). \tag{18}$$

$\Phi^I(x_m, \bar{p}_4R_5x_4 - \bar{p}_5R_4x_5)$ is the discrete version of Φ^I. In continuous limit,

$$X^I(x_m, x_4, x_5) = \int d\theta dp_\theta p_\theta\, e^{ip_\theta(-\sin\theta x_4 + \cos\theta x_5)}\phi^I_{(\theta;p_\theta)}(x_m) = \int d\theta\, \tilde{\Phi}^I(x_m, -\sin\theta x_4 + \cos\theta x_5). \tag{19}$$

However, $\tilde{\Phi}^I$ is not the Φ^I in (15). The latter is

$$\Phi^I(x_m, -\sin\theta x_4 + \cos\theta x_5) = \int dp_\theta\; e^{ip_\theta(-\sin\theta x_4+\cos\theta x_5)}\phi^I_{(\theta;p_\theta)}(x_m) \tag{20}$$

with p_θ left out in the integral. X^I is only the direct superposition of $\tilde{\Phi}^I(\theta)$, which is not the zero mode in $C_\mu(\theta)$ direction. Nevertheless, $\left\{\tilde{\Phi}^I(\theta)\mid\forall\theta\in[0,\pi)\right\}$ and $\left\{\Phi^I(\theta)\mid\forall\theta\in[0,\pi)\right\}$ are equivalent bases, so it is indeed possible to reconstruct X^I from Φ^I.

4.2 The coupling of the selfdual strings and the 3-string junctions

We now have a series of θ-parameterized $5d$ $U(N)$ SYM theories, which is effectively a $6d$ theory with 5 scalars, 3 gauge degrees of freedom and 8 fermionic degrees of freedom. It may at least exhaust the 1/2 BPS field content of the $6d$ $(2,0)$ theory. The next problem is the interaction. Fields belong to the same $5d$ SYM theory have the standard SYM coupling among themselves. It is also necessary to consider the couplings involving fields in different $5d$ SYM theories. Actually, the $6d$ SYM theory could also be decomposed in this way, while the local interactions in the original $6d$ theory induce the couplings among the $5d$ theories labeled by different θ.

To see this coupling more explicitly, we'd better decompose the $6d$ fields into the $4d$ KK modes. For scalars, the decomposition is as that in (17). Similarly, for the $6d$ SYM fields such as the scalars $Y^L(x_m,x_4,x_5)$, $L=6,7,8,9$, there is also

$$Y^L(x_m,x_4,x_5) = \int dp_4dp_5\; e^{i(p_4x_4+p_5x_5)}\varphi^L_{(p_4,p_5)}(x_m). \tag{21}$$

The two-field coupling $Y^LY^{L'}$ gives

$$\int dx_4dx_5\; Y^L_{ij}Y^{L'}_{ji} = \int dp_4dp_5\; \varphi^L_{(p_4,p_5)ij}(x_m)\varphi^{L'}_{(-p_4,-p_5)ji}(x_m), \tag{22}$$

the three-field coupling $Y^LY^{L'}Y^{L''}$ gives

$$\int dx_4dx_5\; Y^L_{ij}Y^{L'}_{jk}Y^{L''}_{ki} = \int dp_4dp_5dq_4dq_5\; \varphi^L_{(p_4,p_5)ij}(x_m)\varphi^{L'}_{(q_4,q_5)jk}(x_m)\varphi^{L''}_{(-p_4-q_4,-p_5-q_5)ki}(x_m), \tag{23}$$

and similarly for the n-field coupling. In the dual $D3$ brane picture, $\varphi^L_{(p_4,p_5)ij}(x_m)$ corresponds to the F-string connecting the i_{th} and the j_{th} $D3$ branes represented by the vector (p_4,p_5) in transverse space. The above coupling is possible because the bound state of the $[i,j]$ (p_4,p_5) F-string and the $[j,k]$ (q_4,q_5) F-string is the $[i,k]$ (p_4+q_4,p_5+q_5) F-string. The conclusion also holds in Coulomb branch. $\varphi^L_{(p_4,p_5)ij}(x_m)$ then corresponds to the F-string represented by the vector $(p_4,p_5,\vec{v}_{ij})$ in transverse space.

$$(p_4,p_5,\vec{v}_{ij}) + (q_4,q_5,\vec{v}_{jk}) = (p_4+q_4,p_5+q_5,\vec{v}_{ik}). \tag{24}$$

On the other hand, for fields in tensor multiplet, such as X^I, the two-field coupling $X^IX^{I'}$ is indeed

$$\int dx_4dx_5\; X^I_{ij}X^{I'}_{ji} = \int dp_4dp_5\; \phi^I_{(p_4,p_5)ij}(x_m)\phi^{I'}_{(-p_4,-p_5)ji}(x_m), \tag{25}$$

but the three-field coupling and the n-field coupling cannot take the similar form as (23). On $D3$ branes, $\phi^I_{(p_4,p_5)ij}(x_m)$ corresponds to the $[i,j]$ (p_4,p_5) string[5]. When $(p_4,p_5)\propto(q_4,q_5)$, the bound state of the $[i,j]$ (p_4,p_5) string and the $[j,k]$ (q_4,q_5) string is still the $[i,k]$ (p_4+q_4,p_5+q_5) string, so, the coupling like (23) is possible. $\phi^I_{(p_4,p_5)ij}(x_m)$ and $\phi^{I'}_{(q_4,q_5)jk}(x_m)$ belong to the same $5d$ SYM theory with $\theta=-\arctan(p_4/p_5)$. However, for unparallel (p_4,p_5) and (q_4,q_5), the bound state will be the 3-string junction other than the single string[6]. The similar problem also exists for the $5d$ massive tensor multiplet. The KK modes in $4d$ could be represented by the $[i,j]$ (p_4,p_5) strings with the fixed p_5 but all possible p_4. (p_4,p_5) and (p_4',p_5) are not parallel unless $p_4=p_4'$. So, if we concentrate on a single kind of the selfdual strings, the theory will be the $5d$ SYM theory; if we consider the selfdual strings with the different orientations, the theory will involve the tensor multiplet, for which, the interaction is not the standard SYM type.

Then the problem reduces to the coupling between $\phi_{(p_4,p_5)ij}(x_m)$ and $\phi'_{(q_4,q_5)jk}(x_m)$ for the unparallel (p_4,p_5) and (q_4,q_5). The bound state of the $[i,j]$ (p_4,p_5) string and the $[j,k]$ (q_4,q_5) string is the 3-string junction other than the traditional $[i,k]$ (P_4,P_5) string. Unlike the $6d$ SYM theory, we now get more states and should also quantize them. A given 3-string junction is characterized by the charge vector $\mathbf{v}_e=(r_4,s_4,t_4)$ and $\mathbf{v}_m=(r_5,s_5,t_5)$, for which, no common divisor exists. r s t are related with the i j k branes, while the rest $N-3$ branes are neglected.

$$r_4+s_4+t_4=r_5+s_5+t_5=0. \tag{26}$$

In x^{45}, $v^{45}_{ij}=2\pi kR'_{45}$, $v^{45}_{jk}=2\pi hR'_{45}$. In transverse space, we may also have $\vec{v}_{ij}$ and $\vec{v}_{jk}$, which will make the string junction massive. The total momentum of the 3-string junction is

$$(P_4,P_5)=(\frac{kr_4-ht_4}{R_4},\frac{-kr_5+ht_5}{R_5})=(p_4+q_4,p_5+q_5). \tag{27}$$

where $(p_4,p_5)=(kr_4/R_4,-kr_5/R_5)=(k\bar{p}_4/R_4,-k\bar{p}_5/R_5)$, $(q_4,q_5)=(-ht_4/R_4,ht_5/R_5)=(h\bar{q}_4/R_4,-h\bar{q}_5/R_5)$. $\bar{p}_4$ and $\bar{p}_5$, $\bar{q}_4$ and $\bar{q}_5$ are not necessarily co-prime now. For $(\bar{p}_4,\bar{p}_5)\propto(\bar{q}_4,\bar{q}_5)$, $(P_4,P_5)\propto(\bar{q}_4,\bar{q}_5)\propto(\bar{p}_4,\bar{p}_5)$, while for the unparallel $(\bar{p}_4,\bar{p}_5)$ and $(\bar{q}_4,\bar{q}_5)$, k and h may generate two dimensional momentum. Especially, if the $SL(2,\mathbf{Z})$ invariant intersection number [16] $I=t_5r_4-t_4r_5=\pm1$, (P_4,P_5) can cover all of $(n/R_4,m/R_5)$; otherwise, it can only cover $(nI/R_4,mI/R_5)$. We will use $(\bar{p}_4,\bar{p}_5)$, $(\bar{q}_4,\bar{q}_5)$ and (P_4,P_5) to denote the 3-string junction. When $R_4,R_5\to\infty$, R_4/R_5 is indefinite, so the 3-string junction is denoted by $(\theta_1,\theta_2,\theta_3)$. For the given $\vec{v}_{ij}$ and $\vec{v}_{jk}$, the 3-string junction exists only when (P_4,P_5) satisfies some particular condition

$$h_{(\bar{p}_4,\bar{p}_5,\bar{q}_4,\bar{q}_5,\vec{v}_{ij},\vec{v}_{jk})}(P_4,P_5)>0, \tag{28}$$

so the 3-string junction cannot have the arbitrary momentum in 45 plane. The selfdual string in 45 plane can only carry the $1d$ transverse momentum. Here, the bound state of two

[5]More accurately, it is the $[i,j]$ $(\bar{p}_4,\bar{p}_5)$ string winding x'^{45} k times. $p_4=k\bar{p}_4/R_4$, $p_5=-k\bar{p}_5/R_5$, $\bar{p}_4$ and $\bar{p}_5$ are co-prime. For simplicity, we just denote it by (p_4,p_5).

[6]The bound state exists only when the $[i,j]$ $[j,k]$ strings have the suitable mass. Here, we just assume so.

unparallel selfdual strings can carry the $2d$ momentum, but this momentum cannot cover the $2d$ space.

Fields arising from the quantization of the 3-string junctions can then be denoted by $\phi_{(\vec{r},\vec{s},\vec{t};P_4,P_5)ijk}(x_m)$ or $\phi_{(\theta_1,\theta_2,\theta_3;P_4,P_5)ijk}(x_m)$ in decompactification limit. $\vec{r}+\vec{s}+\vec{t}=0$. The corresponding $6d$ field is

$$X_{(\vec{r},\vec{s},\vec{t})ijk}(x_m,x_4,x_5)=\sum_{P_4,P_5} e^{i(P_4x_4+P_5x_5)}\phi_{(\vec{r},\vec{s},\vec{t};P_4,P_5)ijk}(x_m), \tag{29}$$

or

$$X_{(\theta_1,\theta_2,\theta_3)ijk}(x_m,x_4,x_5)=\int dP_4dP_5\; e^{i(P_4x_4+P_5x_5)}\phi_{(\theta_1,\theta_2,\theta_3;P_4,P_5)ijk}(x_m). \tag{30}$$

In polar coordinate, (30) could also be written as

$$\begin{aligned}&X_{(\theta_1,\theta_2,\theta_3)ijk}(x_m,\theta,\rho)\\ =\;&\sin(\theta_2-\theta_1)\int dp_{\theta_1}dp_{\theta_2}\; e^{i\rho[\sin(\theta-\theta_1)p_{\theta_1}-\sin(\theta-\theta_2)p_{\theta_2}]}\phi_{(\theta_1,\theta_2,\theta_3;p_{\theta_1},p_{\theta_2})ijk}(x_m).\end{aligned} \tag{31}$$

In terms of k and h, (29) becomes

$$X_{(\vec{r},\vec{s},\vec{t})ijk}(x_m,x_4,x_5)=\sum_{k,h} e^{i[k(\frac{r_4x_4}{R_4}-\frac{r_5x_5}{R_5})-h(\frac{t_4x_4}{R_4}-\frac{t_5x_5}{R_5})]}\phi_{(\vec{r},\vec{s},\vec{t};k,h)ijk}(x_m). \tag{32}$$

In (29)-(32), (P_4,P_5) is in the range specified by (28).

With the fields related with strings as well as the string junctions, we can consider the possible couplings among them. First, the bound state of the $[i,j]$ (r_4,r_5) string and the $[j,k]$ $(-t_4,-t_5)$ string, or the $[j,k]$ (s_4,s_5) string and the $[k,i]$ $(-r_4,-r_5)$ string, or the $[k,i]$ (t_4,t_5) string and the $[i,j]$ $(-s_4,-s_5)$ string is the $[i,j,k]$ 3-string junction with $\mathbf{v}_e=(r_4,s_4,t_4)$ and $\mathbf{v}_m=(r_5,s_5,t_5)$. The momentum of the 3-string junction is the sum of the two individual strings. $2+2\to 3$.

$$\phi_{(r_4,r_5;k)ij}(x_m)\phi'_{(-t_4,-t_5;h)jk}(x_m)\sim\phi''_{(\vec{r},\vec{s},\vec{t};k,h)ijk}(x_m), \tag{33}$$

$$\phi_{(s_4,s_5;h)jk}(x_m)\phi'_{(-r_4,-r_5;-k-h)ki}(x_m)\sim\phi''_{(\vec{r},\vec{s},\vec{t};k,h)ijk}(x_m), \tag{34}$$

$$\phi_{(t_4,t_5;-k-h)ki}(x_m)\phi'_{(-s_4,-s_5;k)ij}(x_m)\sim\phi''_{(\vec{r},\vec{s},\vec{t};k,h)ijk}(x_m), \tag{35}$$

which could be derived from the coupling

$$\int dx_4dx_5\; X_{(r_4,r_5)ij}(x_m,x_4,x_5)X'_{(-t_4,-t_5)jk}(x_m,x_4,x_5)X''_{(\vec{r},\vec{s},\vec{t})ijk}(x_m,x_4,x_5), \tag{36}$$

with

$$X_{(r_4,r_5)ij}(x_m,x_4,x_5)=\sum_k e^{ik(\frac{r_4x_4}{R_4}-\frac{r_5x_5}{R_5})}\phi_{(r_4,r_5;k)ij}(x_m), \tag{37}$$

$$X'_{(-t_4,-t_5)jk}(x_m,x_4,x_5)=\sum_h e^{-ih(\frac{t_4x_4}{R_4}-\frac{t_5x_5}{R_5})}\phi_{(-t_4,-t_5;h)jk}(x_m). \tag{38}$$

Next, consider the bound state of $\phi_{(u_4,u_5;g)li}(x_m)$ or $\phi_{(u_4,u_5;-g)il}(x_m)$ and $\phi'_{(\vec{r},\vec{s},\vec{t};k,h)ijk}(x_m)$. If $(u_4,u_5)=(r_4,r_5)$ or $(u_4,u_5)=(-r_4,-r_5)$, the bound state will still be the 3-string junction $\phi''_{(\vec{r},\vec{s},\vec{t};k+g,h)ljk}(x_m)$, $2+3\to 3$; otherwise, it is a 4-string junction, $2+3\to 4$. The situation is similar if i is replaced by j or k. The $2+3\to 3$ type relation may give the couplings like

$$X_{li}X'_{ijk}X''_{ljk}, \quad X_{il}X'_{ijk}X''_{ljk}, \quad X_{li}X'_{ijk}X''_{kn}X'''_{ljn} \tag{39}$$

and so on.

The $2+2\to 2$, $2+3\to 3$ couplings could be realized as the matrix multiplication. Moreover, they can also be visualized as the junction of two 2-boundary-$M2$'s and the junction of one 2-boundary-$M2$ and one 3-boundary-$M2$ respectively. Therefore, they are more reasonable than the couplings like $2+2\to 3$ and $2+3\to 4$.

4.3 The multiplet structure of the 3-string junctions

We now have two sets of fields $f_{(r_4,r_5)ij}(x_m,x_4,x_5)$ and $g_{(\vec{r},\vec{s},\vec{t})ijk}(x_m,x_4,x_5)$, or alternatively, $f_{(\theta)ij}(x_m,\alpha,\rho)$ and $g_{(\theta_1,\theta_2,\theta_3)ijk}(x_m,\alpha,\rho)$. $f_{(r_4,r_5)ij}(x_m,x_4,x_5)$ is translation invariant along the (r_5R_4,r_4R_5) direction. It is the previously discussed field satisfying the constraint $C_\mu(\theta)\partial^\mu f_{(\theta)ij}(x_m,\alpha,\rho)=0$. On the other hand, $g_{(\theta_1,\theta_2,\theta_3)ijk}(x_m,\alpha,\rho)$ is a $6d$ field without the constraint[7]. $f_{(\theta_1)}$, $f_{(\theta_2)}$ and $f_{(\theta_3)}$ may couple with each other through $g_{(\theta_1,\theta_2,\theta_3)}$.

$f_{(\theta)ij}$ is a vector multiplet composed by the scalars $\Phi^I_{(\theta)}$, the vector $A_{(\theta)\mu}$ and the spinor $\eta_{(\theta)}$ with the scaling dimensions 1, 1 and 3/2 respectively, coming from the $C_\mu(\theta)$ direction integration of the scalars X^I, the 2-form $B_{\mu\nu}$ and the spinor Ψ with the scaling dimensions 2, 2 and 5/2. As a $6d$ vector, $C^\mu(\theta)A_{(\theta)\mu}=0$.

The field content of $g_{(\theta_1,\theta_2,\theta_3)ijk}$ can be reconstructed from the $4d$ KK mode. The KK compactification of $g_{(\theta_1,\theta_2,\theta_3)ijk}$ on $x_4\times x_5$ gives the $4d$ field $\phi_{(\vec{r},\vec{s},\vec{t};P_4,P_5)ijk}$, which, in $4d$ SYM theory, is related to the 3-string junction with the charge vector $\mathbf{v}_e=(r_4,s_4,t_4)$ and $\mathbf{v}_m=(r_5,s_5,t_5)$, having the total mass P_5 and the total electric charge P_4. The multiplet structure of $\phi_{(\vec{r},\vec{s},\vec{t};P_4,P_5)ijk}$ is $V_4\otimes V_{in}$, where V_4 is the vector supermultiplet coming from the free center-of-mass part, V_{in} is the internal part determined by $\mathbf{v}_e$ and $\mathbf{v}_m$. For $\mathbf{v}_e=(r_4,s_4,t_4)$, $\mathbf{v}_m=(1,0,-1)$, $V_{in}=[|s_4|/2]\oplus[|s_4|/2-1/2]\oplus[|s_4|/2-1/2]\oplus[|s_4|/2-1]$, giving a total of $4|s_4|$ states [14]. As the string web, it has $E_{ext}=3$ external points and $F_{int}=|s_4|$ internal points [16]. If $\phi_{(\vec{r},\vec{s},\vec{t};P_4,P_5)ijk}$ is lifted into the $6d$ field $g_{(\vec{r},\vec{s},\vec{t})ijk}$, $V_4\otimes V_{in}$ will become $T_6\otimes V_{in}$, with T_6 the tensor supermultiplet from the center-of-mass part. So, $g_{(\vec{r},\vec{s},\vec{t})ijk}$ at least contains a tensor multiplet factor.

It is difficult to determine V_{in} in decompactification limit. The simplest situation is $|s_4|=1$ with one internal point, and then $V_{in}=[1/2]\oplus[0]\oplus[0]$. Recall that for 1/2 BPS states with the degeneracy of 2^4, we have the $6d$ $(2,0)$ tensor multiplet T_6, whose KK modes along x_5 are the $5d$ massless vector multiplet V_5 and the $5d$ massive $(2,0)$ tensor multiplets T_5. The KK modes of V_5 and T_5 on x_4 are the $4d$ vector multiplets V_4. The massless limit

[7] (28) gives a restriction on the range of k and h in (32). Especially, if $k=0$ or $h=0$, $g_{(\theta_1,\theta_2,\theta_3)ijk}$ is also translation invariant along one direction, as we will see later.

of the $5d$ massive tensor multiplet is the $5d$ massless vector multiplet. For 1/4 BPS states, in $4d$, we get $V_4 \otimes ([1/2] \oplus [0] \oplus [0])$. The $V_4 \otimes [1/2]$ part gives

j	3/2	1	1/2	0	-1/2	-1	-3/2
Degeneracy	1	4	7	8	7	4	1

which, when combines with the rest two V_4, could be organized into 1 spin-3/2 fermion, 6 vectors, 14 spin-1/2 fermions and 14 scalars, forming the massive representation of the $4d$ $\mathcal{N}=4$ superalgebra with 2^6 states. In massless limit, the bosonic part of $V_4 \otimes ([1/2] \oplus [0] \oplus [0])$ is composed by 6 vectors and 20 scalars, with each vector containing two degrees of freedom. V_4, when lifted to $5d$ with $P_5 = 0$ or $P_5 \neq 0$, becomes V_5 or T_5. The lifted $V_4 \otimes ([1/2] \oplus [0] \oplus [0])$ could be naively denoted by $V_5 \otimes ([1/2] \oplus [0] \oplus [0])$ and $T_5 \otimes ([1/2] \oplus [0] \oplus [0])$, which are all complex now. Actually, one $V_4 \otimes ([1/2] \oplus [0] \oplus [0])$ only gives the 3-string junction with one possible orientation; if the other orientation is taken into account, we will also get $2^6 \times 2 = 2^7$ states. The field content of $V_5 \otimes ([1/2] \oplus [0] \oplus [0])$ could be organized into 1 spin-3/2 fermion, 6 vectors, 13 spin-1/2 fermions and 14 scalars, while the field content of $T_5 \otimes ([1/2] \oplus [0] \oplus [0])$ could be organized into 2 selfdual tensors, 1 spin-3/2 fermion, 4 vectors, 13 spin-1/2 fermions and 10 scalars. $T_6 \otimes ([1/2] \oplus [0] \oplus [0])$ and $T_5 \otimes ([1/2] \oplus [0] \oplus [0])$ have the same field content, forming the $6d$ massless $(2,1)$ multiplet and the $5d$ massive $(2,1)$ multiplet respectively. The $5d$ massive selfdual tensors and the $5d$ massive vectors, containing 3 and 4 degrees of freedom, become the $6d$ massless selfdual tensors and the $6d$ massless vectors, still with 3 and 4 degrees of freedom.

$T_6 \otimes ([1/2] \oplus [0] \oplus [0])$ compactified on x_5 gives $V_5 \otimes ([1/2] \oplus [0] \oplus [0])$ and $T_5 \otimes ([1/2] \oplus [0] \oplus [0])$, which, when further compactified on x_4, becomes $V_4 \otimes ([1/2] \oplus [0] \oplus [0])$. Just as V_5 is the massless limit of T_5, $V_5 \otimes ([1/2] \oplus [0] \oplus [0])$ could also be taken as the massless limit of $T_5 \otimes ([1/2] \oplus [0] \oplus [0])$. The 2 massive $5d$ selfdual tensors become 2 massless $5d$ vectors, while the 4 massive $5d$ vectors become 4 massless $5d$ vectors plus 4 scalars. $T_6 \otimes ([1/2] \oplus [0] \oplus [0])$, $T_5 \otimes ([1/2] \oplus [0] \oplus [0])$ and $V_5 \otimes ([1/2] \oplus [0] \oplus [0])$ are all complex, so the total states for each are 2^7 other than 2^6. Each multiplet will form the $N \times N \times N$ or $\bar{N} \times \bar{N} \times \bar{N}$ representation of $U(N)$, so they cannot be real, as the fields in adjoint representation do.

4.4 The coupling among the vector multiplet and the $(2,1)$ multiplet

$g_{(\theta_1,\theta_2,\theta_3)ijk}$ is the $(2,1)$ multiplet composed by the scalars $X_{(\theta_1,\theta_2,\theta_3)}$, the vectors $V_{(\theta_1,\theta_2,\theta_3)}$, the 2-forms $B_{(\theta_1,\theta_2,\theta_3)}$, the spin-1/2 fermions $\Psi_{(\theta_1,\theta_2,\theta_3)}$ and the spin-3/2 fermions $\eta_{(\theta_1,\theta_2,\theta_3)}$. In principle, the $6d$ $(2,0)$ theory can only contain the $(2,0)$ tensor multiplet, but now, the $(2,1)$ multiplet is also added. There will be the couplings between the $(2,1)$ multiplet and the vector multiplet arising from the reduction of the tensor multiplet along a particular direction. The incorporation of the $(2,1)$ multiplet into the scattering amplitude is also discussed in [15] for $M5$ compactified on S^1. It was shown that the $BB'A$ coupling is one

of the possibilities. B and B' are the 2-forms in $5d$ massive $(2,1)$ multiplet, while A is the zero mode vector in $5d$. In the following, we will only discuss X, B and Ψ with the scaling dimensions 2, 2, 5/2 respectively, neglecting V and η. The transverse indices of X, B and Ψ are dropped for simplicity, although X, B and Ψ are not the R-symmetry singlet.

Let us consider the possible dimension six couplings for these fields. For two-field couplings, there are

$$\partial X \partial X, \;\; \partial X_{(\theta_1,\theta_2,\theta_3)} \partial X^*_{(\theta_1,\theta_2,\theta_3)}, \;\; \bar{\Psi}\partial\Psi, \;\; \bar{\Psi}_{(\theta_1,\theta_2,\theta_3)} \partial \Psi_{(\theta_1,\theta_2,\theta_3)}, \;\; \partial B \partial B, \;\; \partial B_{(\theta_1,\theta_2,\theta_3)} \partial B^*_{(\theta_1,\theta_2,\theta_3)}. \tag{40}$$

X Ψ B compose a $6d$ $(2,0)$ tensor multiplet in adjoint representation of $U(N)$, which is equivalent to $\Phi_{(\theta)}$ $A_{(\theta)}$ $\eta_{(\theta)}$ with all θ included. We do not have terms like $X_{ij}X'_{jk}X''_{ki}$, but the two-field couplings like $X_{ij}X'_{ji}$ are allowed. The tensor multiplet representation works well in free theory. The possible three-field couplings are

$$A_{\theta_a} X_{(\theta_1,\theta_2,\theta_3)} \partial X^*_{(\theta_1,\theta_2,\theta_3)}, \; A_{\theta_a} \bar{\Psi}_{(\theta_1,\theta_2,\theta_3)} \Psi_{(\theta_1,\theta_2,\theta_3)}, \; \Phi_{\theta_a} \bar{\Psi}_{(\theta_1,\theta_2,\theta_3)} \Psi_{(\theta_1,\theta_2,\theta_3)}, \; A_{\theta_a} B_{(\theta_1,\theta_2,\theta_3)} \partial B^*_{(\theta_1,\theta_2,\theta_3)}, \tag{41}$$

where $a = 1, 2, 3$. The possible four-field couplings are

$$\begin{aligned} & A_{\theta_a} X_{(\theta_1,\theta_2,\theta_3)} A_{\theta_b} X^*_{(\theta_1,\theta_2,\theta_3)}, \qquad \Phi_{\theta_a} X_{(\theta_1,\theta_2,\theta_3)} \Phi_{\theta_b} X^*_{(\theta_1,\theta_2,\theta_3)}, \\ & A_{\theta_a} B_{(\theta_1,\theta_2,\theta_3)} A_{\theta_b} B^*_{(\theta_1,\theta_2,\theta_3)}, \qquad \Phi_{\theta_a} B_{(\theta_1,\theta_2,\theta_3)} \Phi_{\theta_b} B^*_{(\theta_1,\theta_2,\theta_3)}, \end{aligned} \tag{42}$$

with $a, b = 1, 2, 3$. Based on the above couplings, the nonabelian generalization of $H_{\mu\nu\lambda}$ can then be defined as

$$H_{ijk} = dB_{ijk} + A_i^l \wedge B_{ljk} + A_j^m \wedge B_{imk} + A_k^n \wedge B_{ijn}, \tag{43}$$

with $H \sim H_{(\theta_1,\theta_2,\theta_3)\mu\nu\lambda}$, $A_i^l \sim A^l_{(\theta_1)\mu i}$, $A_j^m \sim A^m_{(\theta_2)\mu j}$, $A_k^n \sim A^n_{(\theta_3)\mu k}$, $B \sim B_{(\theta_1,\theta_2,\theta_3)\mu\nu}$.

Fermions may get mass through the Yukawa coupling $\Phi_{\theta_a} \bar{\Psi}_{(\theta_1,\theta_2,\theta_3)} \Psi_{(\theta_1,\theta_2,\theta_3)}$. In order to compare with the 3-string junctions in $4d$ SYM theory, we will use $(\vec{r}, \vec{s}, \vec{t})$ instead of $(\theta_1, \theta_2, \theta_3)$. Consider $\Psi_{(\vec{r},\vec{s},\vec{t})ijk}$ and $\Phi^I_{(\vec{u};lm)\mu}$. The vacuum expectation value of X^I is $\bar{X}^I_{lm} = v^I_m \delta_{lm}$, then the induced vacuum expectation value for Φ^I_μ is $\bar{\Phi}^I_{(\vec{u};lm)\mu} = \tilde{u}_\mu v^I_m \delta_{lm}$. Similar with the equation for fermions in [9],

$$\Gamma^\mu D_\mu \Psi_A + X^I_C C^\nu_B \Gamma_\nu \Gamma^I \Psi_D f^{CDB}{}_A = 0, \tag{44}$$

we may have

$$\begin{aligned} & i\Gamma^0\Gamma^\mu\Gamma_I[\bar{\Phi}^I_{(\vec{r};il)\mu}\Psi_{(\vec{r},\vec{s},\vec{t})ljk} + \bar{\Phi}^I_{(\vec{s};jl)\mu}\Psi_{(\vec{r},\vec{s},\vec{t})ilk} + \bar{\Phi}^I_{(\vec{t};kl)\mu}\Psi_{(\vec{r},\vec{s},\vec{t})ijl}] \\ = \; & i\Gamma^0\Gamma^\mu\Gamma_I[\tilde{r}_\mu v^I_i \delta_{il}\Psi_{(\vec{r},\vec{s},\vec{t})ljk} + \tilde{s}_\mu v^I_j \delta_{jl}\Psi_{(\vec{r},\vec{s},\vec{t})ilk} + \tilde{t}_\mu v^I_k \delta_{kl}\Psi_{(\vec{r},\vec{s},\vec{t})ijl}] \\ = \; & i\Gamma^0\Gamma^\mu\Gamma_I(\tilde{r}_\mu v^I_i + \tilde{s}_\mu v^I_j + \tilde{t}_\mu v^I_k)\Psi_{(\vec{r},\vec{s},\vec{t})ijk} = M\Psi_{(\vec{r},\vec{s},\vec{t})ijk}, \end{aligned} \tag{45}$$

where in the last step, we assume $\Psi_{lmn} = 0$ for $l, m, n \neq i, j, k$ so that Ψ is a generator with the index $[i, j, k]$.

$$M = i\Gamma^0\Gamma^\mu\Gamma_I(\tilde{r}_\mu v^I_{ij} - \tilde{t}_\mu v^I_{jk}) = i\Gamma^0\Gamma^\mu\Gamma_I(\tilde{s}_\mu v^I_{jk} - \tilde{r}_\mu v^I_{ki}) = i\Gamma^0\Gamma^\mu\Gamma_I(\tilde{t}_\mu v^I_{ki} - \tilde{s}_\mu v^I_{ij}). \tag{46}$$

The $[i,j,k]$ $(\vec{r},\vec{s},\vec{t})$ string is the bound state of the $[i,j]$ $(\vec{r})$ and $[j,k]$ $(-\vec{t})$ strings or the $[j,k]$ $(\vec{s})$ and $[k,i]$ $(-\vec{r})$ strings or the $[k,i]$ $(\vec{t})$ and $[i,j]$ $(-\vec{s})$ strings. In (46), the mass of the bound state is expressed in terms of the component strings. $\tilde{r}_4 = 2\pi r_5 R_4$, $\tilde{r}_5 = 2\pi r_4 R_5$, $\tilde{r}_\mu = 0$, for $\mu = 0,1,2,3$, so

$$i\Gamma^0\Gamma^\mu\Gamma_I\tilde{r}_\mu v_i^I = i\Gamma^0\Gamma^4\Gamma_I 2\pi r_5 R_4 v_i^I + i\Gamma^0\Gamma^5\Gamma_I 2\pi r_4 R_5 v_i^I, \tag{47}$$

and similarly for $\tilde{s}_\mu$ and $\tilde{t}_\mu$. As a result,

$$M = i\Gamma^0\Gamma^4\Gamma_I(\tilde{r}_4 v_i^I + \tilde{s}_4 v_j^I + \tilde{t}_4 v_k^I) + i\Gamma^0\Gamma^5\Gamma_I(\tilde{r}_5 v_i^I + \tilde{s}_5 v_j^I + \tilde{t}_5 v_k^I) = i\Gamma^0\Gamma^4\Gamma_I Q_M^I + i\Gamma^0\Gamma^5\Gamma_I Q_E^I. \tag{48}$$

where Q_E^I and Q_M^I are the electric and the magnetic charge vectors in $4d$ SYM theory.

$$M^2 = |\vec{Q}_E|^2 + |\vec{Q}_M|^2 + \Gamma_I\Gamma_J\Gamma^4\Gamma^5(Q_M^I Q_E^J - Q_M^J Q_E^I). \tag{49}$$

The third term is a matrix, nevertheless,

$$\sqrt{[\Gamma_I\Gamma_J\Gamma^4\Gamma^5(Q_M^I Q_E^J - Q_M^J Q_E^I)]^2} = 2|\vec{Q}_E \times \vec{Q}_M| \tag{50}$$

The above result can be compared with the mass of the 3-string junctions in $4d$ SYM theory, which is

$$Z_+^2 = |\vec{Q}_E|^2 + |\vec{Q}_M|^2 + 2|\vec{Q}_E \times \vec{Q}_M|. \tag{51}$$

The mass term together with $i\Psi^+\Gamma_\mu\partial^\mu\Psi$ gives the energy

$$E = \Gamma_0\Gamma_\mu p^\mu + i\Gamma^0\Gamma^4\Gamma_I Q_M^I + i\Gamma^0\Gamma^5\Gamma_I Q_E^I, \tag{52}$$

where $\mu = 1,2,3,4,5$, $\Gamma_\mu^+ = -\Gamma_\mu$.

$$E^2 = |\vec{p}|^2 + |\vec{Q}_E|^2 + |\vec{Q}_M|^2 + \Gamma_I\Gamma_J\Gamma^4\Gamma^5(Q_M^I Q_E^J - Q_M^J Q_E^I) + 2i\Gamma_I(Q_M^I p^4 + Q_E^I p^5). \tag{53}$$

(53) can be rewritten as

$$\begin{aligned} E^2 &= p_a p_a + (Q_E^{45}Q_E^{45} + Q_M^{45}Q_M^{45}) + (Q_E^I Q_E^I + Q_M^I Q_M^I) \\ &+ \Gamma_I\Gamma_J\Gamma^4\Gamma^5(Q_M^I Q_E^J - Q_M^J Q_E^I) + 2i\Gamma_I(Q_E^{45}Q_M^I - Q_E^I Q_M^{45}), \end{aligned} \tag{54}$$

where $a = 1,2,3$. $Q_E^{45} = p^4$, $Q_M^{45} = -p^5$. p_4 and p_5 enter the energy formula as another charge vector Q_E^{45} and Q_M^{45}. p^1, p^2 and p^3 appear as the normal transverse momentum. In $4d$ SYM theory, with v_i^{45} and v_i^I turned on, the energy of the 3-string junction carrying the transverse momentum (p^1,p^2,p^3) is consistent with (54). The above result can be compared with the $6d$ SYM theory, for which,

$$E = \Gamma_0\Gamma_\mu p^\mu + \Gamma_0\Gamma_I v_{ij}^I, \tag{55}$$

so

$$E^2 = p_\mu p_\mu + v_{ij}^I v_{ij}^I, \tag{56}$$

which is the energy of a particle with the rest mass $\sqrt{v^I_{ij} v^I_{ij}}$ carrying the $5d$ momentum p^μ. Now, we have different Dirac operator, giving rise to a dispersion relation different from the standard $\sqrt{m^2+p^2}$ type. $\sqrt{m^2+p^2}$ is the dispersion relation for a Lorentz invariant theory. The 3-string junctions breaks the $SO(5,1)$ symmetry into $SO(3,1)$.

A_{N-1} $6d$ $(2,0)$ theory compactified on a Riemann surface Σ_g with the genus $g>1$ could be decomposed into the T_N part and the I_N part. Each T_N part has the $SU(N)^3$ symmetry, while each I_N part gives a $SU(N)$ gauge group [24, 25]. Still, there are two sets of fields with the index $[i,j,k]$ and $[i,j]$ which may couple with each other, quite like what we have discussed above. This is not accidental. The 3-string junction on $D3$, when lifted to M theory, corresponds to $M2$ with three boundaries, which may be denoted by $M2(C_1,C_2,C_3)$, with $C_1 \sim \vec{r}$, $C_2 \sim \vec{s}$, $C_3 \sim \vec{t}$ [26]. $M2$ with two boundaries is $M2(C)$. $M2(C_1,C_2,C_3)$ and $M2(C)$ may couple at the boundary as long as $C=C_1$, or $C=C_2$, or $C=C_3$, while the product is still $M2(C_1,C_2,C_3)$. Likewise, the T_N part of the Riemann surface offers the nontrivial 1-cycles $[C_1]$, $[C_2]$, $[C_3]$ for $M2$ to end. $[C_1]+[C_2]+[C_3]=0$. Each $M2([C_1],[C_2],[C_3])$ can only couple with the adjacent $M2([C_1])$, $M2([C_2])$, and $M2([C_3])$.

The Σ_g theory has $3(g-1)$ $SU(N)$ gauge groups associated with the $3(g-1)$ 1-cycles. Similarly, the $6d$ $(2,0)$ theory may contain a series of $SU(N)$ groups associated with the selfdual strings labeled by θ. A different way to decompose Σ_g will give a different set of $3(g-1)$ 1-cycles, for which, the corresponding $4d$ theory is S-dual to the previous one. Likewise, selfdual strings parallel to a different plane may give a different $6d$ theory which is U-dual to the original one. The situation is different for the $6d$ SYM theory, in which, there is only one gauge group. Even if the $6d$ SYM theory is compactified on a Riemann surface with $g>1$, there is still only one gauge group, while the resulting $4d$ theory is unique without the dual version. The reason is that the basic excitations on $M5$ is line-like, while the basic excitations on $D5$ is point-like. $M5$ compactified on Σ_g has the richer structure than $D5$.

4.5 The situation on coincident $M5$ branes

In above discussion, we didn't pay too much attention to the condition (28). For the given v^I_i and $(\vec{r},\vec{s},\vec{t})$, the allowed (P_4,P_5) are not arbitrary. Especially, when $v^I_{ij}=0$, $\forall\, i,j$, no (P_4,P_5) can satisfy (28). Nevertheless, when $P_4=0$ or $P_5=0$, the equality can be saturated, while the bound states are at the threshold or just decay. If they do not decay, then (31) and (32) should be replace by

$$X_{(\theta_1,\theta_2,\theta_3)ijk}(x_m,\sin(\theta-\theta_1)\rho)=\sin(\theta_2-\theta_1)\int dp_{\theta_1}\; e^{ip_{\theta_1}\rho\sin(\theta-\theta_1)}\phi_{(\theta_1,\theta_2,\theta_3;p_{\theta_1},0)ijk}(x_m) \quad (57)$$

and

$$X_{(\vec{r},\vec{s},\vec{t})ijk}(x_m,r_4x_4R_5-r_5x_5R_4)=\sum_k\; e^{ik(\frac{r_4x_4}{R_4}-\frac{r_5x_5}{R_5})}\phi_{(\vec{r},\vec{s},\vec{t};k,0)ijk}(x_m). \quad (58)$$

(57) and (58) are translation invariant along the θ_1 direction and the (r_5R_4,r_4R_5) direction respectively. They are the zero mode of the original $6d$ field (31) and (32) along the θ_1 and

the (r_5R_4, r_4R_5) directions. $\phi_{(\vec{r},\vec{s},\vec{t};k,0)ijk}(x_m)$ is the $4d$ 1/4 BPS field in $V_4 \otimes V_{in}$ multiplet. Summing over all possible k will give a $5d$ field in $V_5 \otimes V_{in}$ multiplet. (57) and (58) are in the $V_5 \otimes V_{in}$ multiplet. They are actually the bound state of the $[i, j]$ (r_4, r_5) string with momentum $(kr_4/R_4, -kr_5/R_5)$ and the $[j, k]$ $(-t_4, -t_5)$ string with momentum $(0, 0)$. As is mentioned before, the $4d$ $(0, 0)$ mode of the $6d$ field is unique, so $(-t_4, -t_5)$ should be fixed, while the field in (57) and (58) could simply be denoted by $X_{(\theta_1)ijk}(x_m, \sin(\theta - \theta_1)\rho)$ and $X_{(\vec{r})ijk}(x_m, r_4x_4R_5 - r_5x_5R_4)$. Although $X_{(\theta_1)ijk}$ or $X_{(\vec{r})ijk}$ is a $5d$ field, with all θ_1 or (r_4, r_5) included, the $6d$ field can be recovered again.

$$\phi_{(\vec{r};k)ijk}(x_m)\phi'_{(\vec{r};g)li}(x_m) \sim \phi''_{(\vec{r};k+g)ljk}(x_m). \tag{59}$$

$X_{(\vec{r})ijk}$ or $X_{(\theta_1)ijk}$ can only couple with $X_{(\vec{r})li}$ or $X_{(\theta_1)li}$. Both of them are translation invariant along the same direction, so the coupling is still 5 dimensional. Now, we have $f_{ij}(\theta, x_\mu)$ together with $g_{ijk}(\theta, x_\mu)$ subject to the constraints $C_\mu(\theta)\partial^\mu f_{ij}(\theta, x_\mu) = 0$ and $C_\mu(\theta)\partial^\mu g_{ijk}(\theta, x_\mu) = 0$, giving rise to the $6d$ fields. Fields related with different θ cannot couple with each other. It must be admitted that such scenario is not quite interesting.

5 The momentum-carrying BPS states in $5d$ SYM theory

Until now, all of the discussions are carried out in $6d$ theory's framework, in which the KK modes are fields. The $6d$ tensor multiplet field compactified on x_5 gives the $5d$ massless vector multiplet field and a tower of $5d$ massive tensor multiplet fields. As the zero mode, the $5d$ SYM field must have the vanishing Pontryagin number. However, the generic configurations of the $5d$ SYM theory on R^4 can carry the arbitrary Pontryagin number k, while the quantization of the configurations with the nonzero k gives the $5d$ massive tensor multiplets. So the full $5d$ SYM theory contains the complete KK modes and may give another definition of the $6d$ $(2, 0)$ theory [10, 11][8].

The field configurations in SYM theories are classified by the the boundary topology. For $5d$ SYM theory, the boundary configurations are characterized by $\Pi_3(SU(N)) \cong \mathbf{Z}$ with $k \in \mathbf{Z}$ the winding number. Configurations with the same k could be continuously deformed into each other. Especially, when $k = 0$, fields could be continuously deformed to zero. The sector with the given k corresponds to the KK mode with $P_5 = k/R_5$. The energy is bounded by

$$E \geq |k|/R_5. \tag{60}$$

The equality holds for configurations representing the localized k/R_5 mode which have the zero average momentum in 1234 space. The path integral covers all configurations, so the complete $5d$ SYM theory is intrinsically a $6d$ theory. Since the configuration only carries the chargeless P_5 momentum, there might be some kind of confinement happen.

[8] See [27] for a further evidence on the finiteness of the $5d$ SYM theory.

In this section, we will discuss the generic BPS states in $5d$ SYM theory, which are in one-to-one correspondence with the previous mentioned selfdual strings and the string junctions. We will also show that the selfdual strings carrying the longitudinal momentum have the N^3 scaling.

5.1 BPS states in $5d$ SYM theory

The field content of the $5d$ $\mathcal{N}=2$ $U(N)$ SYM theory consists of a vector A_μ with $\mu = 0,1,2,3,4$, five scalars X^I with $I=6,7,8,9,10$ and fermions Ψ. x_5 is the extra dimension associated with M-theory. The action is

$$\begin{aligned} S &= -\frac{1}{g_{YM}^2}\int d^5x \, \mathrm{tr}\Big(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu X^I D^\mu X^I - \frac{i}{2}\bar{\Psi}\Gamma^\mu D_\mu \Psi \\ &\qquad + \frac{1}{2}\bar{\Psi}\Gamma^5\Gamma^I[X^I,\Psi] - \frac{1}{4}\sum_{I,J}[X^I,X^J]^2\Big) \ , \end{aligned} \tag{61}$$

where $D_\mu X^I = \partial_\mu X^I - i[A_\mu, X^I]$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$. For time-independent bosonic solutions with a single non-vanishing scalar field X^6, the associated energy is

$$E = \frac{1}{g_{YM}^2}\int d^4x \, \mathrm{tr}\Big[\frac{1}{4}F_{ij}F_{ij} + \frac{1}{2}F_{0i}F_{0i} + \frac{1}{2}D_iX^6D_iX^6\Big], \tag{62}$$

where $i=1,2,3,4$. For an arbitrary vector C_i with $|C|=1$, E could be rewritten as

$$\begin{aligned} E &= \frac{1}{g_{YM}^2}\int d^4x \, \mathrm{tr}\Big[\frac{1}{2}(F_{0i} - \sin\theta C_k F_{ik} + \cos\theta D_i X^6)^2 \\ &\qquad + \frac{1}{2}(\frac{1}{2}C_k\varepsilon_{ilmk}F_{lm} \pm \cos\theta C_k F_{ik} \pm \sin\theta D_i X^6)^2 \\ &\qquad + \sin\theta\,(F_{0i}C_kF_{ik} \mp \frac{1}{2}C_k\varepsilon_{ilmk}F_{lm}D_iX^6) \\ &\qquad + \cos\theta\,(\mp\frac{1}{8}\varepsilon_{iklm}F_{ik}F_{lm} - F_{0i}D_iX^6)\Big] \ . \end{aligned} \tag{63}$$

Note that

$$\begin{aligned} &P_k = -\frac{1}{g_{YM}^2}\int d^4x \, \mathrm{tr}(F_{0i}F_{ik}), \quad Q_{Mk} = Z_k^6 = -\frac{1}{2g_{YM}^2}\int d^4x \, \mathrm{tr}(\varepsilon_{iklm}F_{lm}D_iX^6), \\ &P_5 = -\frac{1}{8g_{YM}^2}\int d^4x \, \mathrm{tr}(\varepsilon_{iklm}F_{lm}F_{ik}), \quad Q_E = Z_5^6 = \frac{1}{g_{YM}^2}\int d^4x \, \mathrm{tr}(F_{0i}D_iX^6), \end{aligned} \tag{64}$$

So

$$E \geq \sin\theta C_k(-P_k \pm Q_{Mk}) + \cos\theta(\pm P_5 - Q_E) \geq Max(Z_+, Z_-), \tag{65}$$

where

$$Z_\pm = \Big[(C_kP_k \pm C_kQ_{Mk})^2 + (P_5 \pm Q_E)^2\Big]^{\frac{1}{2}}. \tag{66}$$

If $Z_+ \geq Z_-$, $E = Z_+$ for

$$F_{0i} = \sin\theta C_k F_{ik} - \cos\theta D_i X^6, \tag{67}$$

$$\frac{1}{2}C_k\varepsilon_{ilmk}F_{lm} = \cos\theta C_k F_{ik} + \sin\theta D_i X^6. \tag{68}$$

If $Z_+ \leq Z_-$, $E = Z_-$ for

$$F_{0i} = \sin\theta C_k F_{ik} - \cos\theta D_i X^6, \tag{69}$$

$$\frac{1}{2}C_k\varepsilon_{ilmk}F_{lm} = -\cos\theta C_k F_{ik} - \sin\theta D_i X^6. \tag{70}$$

In both cases,

$$E = \frac{1}{g_{YM}^2}\int d^4x \, \mathrm{tr}\Big[(C_k F_{ik})^2 + (D_i X^6)^2\Big]. \tag{71}$$

Moreover, if $\theta \neq 0$, from (67-70), we also have $C_i D_i X^6 = C_i F_{0i} = 0$. For simplicity, in the following, we will only consider the case with $Z_+ \geq Z_-$. The situation with $Z_+ \leq Z_-$ is similar.

Without loss of generity, let $C_k = \delta_k^4$, then (67) and (68) become

$$F_{0i} = \sin\theta F_{i4} - \cos\theta D_i X^6, \tag{72}$$

$$\frac{1}{2}\varepsilon_{ilm4}F_{lm} = \cos\theta F_{i4} + \sin\theta D_i X^6. \tag{73}$$

$$E = \Big[(P_4 + Q_{M4})^2 + (P_5 + Q_E)^2\Big]^{\frac{1}{2}}. \tag{74}$$

For $\theta \neq 0$, $F_{04} = D_4 X^6 = 0$. When $\theta = 0$, (72)-(74) reduce to

$$F_{0i} = -D_i X^6, \qquad \frac{1}{2}\varepsilon_{ilm4}F_{lm} = F_{i4}. \tag{75}$$

$$E = |P_5 + Q_E|. \tag{76}$$

These are the equations for the dyonic instantons discussed in [11]. $F_{04} = D_4 X^6 = 0$ is not necessary. If is imposed, the original $SO(4)$ symmetry will be broken to $SO(3)$. $P_4 \neq 0$, $Q_{M4} \neq 0$, but $P_4 + Q_{M4} = 0$. When $\theta = \pi/2$,

$$F_{0i} = F_{i4}, \qquad \frac{1}{2}\varepsilon_{ilm4}F_{lm} = D_i X^6. \tag{77}$$

$$E = |P_4 + Q_{M4}|. \tag{78}$$

The solution describes the monopole string extending along the x_4 direction, carrying momentum P_4. $P_5 \neq 0$, $Q_E \neq 0$, but $P_5 + Q_E = 0$.

For the time-independent bosonic solutions with $C_k = \delta_k^4$, the supersymmetry transformation becomes

$$\begin{aligned}
\delta_\epsilon \Psi &= \frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu}\Gamma_5\epsilon + D_\mu X^6 \Gamma^\mu \Gamma^6 \epsilon \\
&= D_a X^6 \Gamma_a (\Gamma^6 + \cos\theta\Gamma_{05} + \sin\theta\Gamma_{123}\Gamma_5)\epsilon \\
&\quad + F_{a4}\Gamma_a(\Gamma_{45} - \sin\theta\Gamma_{05} + \cos\theta\Gamma_{123}\Gamma_5)\epsilon,
\end{aligned} \tag{79}$$

where $a = 1, 2, 3$. $D_4X^6 = F_{04} = 0$ is imposed. $\delta_\epsilon\Psi = 0$, ϵ should satisfy

$$(1 + \cos\theta\Gamma_{056} - \sin\theta\Gamma_{046})\epsilon = 0, \tag{80}$$

$$(1 + \sin\theta\Gamma_{04} - \cos\theta\Gamma_{05})\epsilon = 0, \tag{81}$$

in which $\Gamma_{012345}\epsilon = \epsilon$ is used. The solution is 1/4 BPS. For $\theta = 0$, we have $\epsilon = -\Gamma_{056}\epsilon = \Gamma_{05}\epsilon$, which are the supersymmetries preserved by dyonic instantons [11]. For $\theta = \pi/2$, $\epsilon = \Gamma_{046}\epsilon = -\Gamma_{04}\epsilon$, which are the supersymmetries preserved by the monopole strings extending along x_4 carrying momentum P_4. If D_aX^6 and F_{a4} are not independent, for example, $D_aX^6 = \sin\theta D_a\Phi$ and $F_{a4} = \cos\theta D_a\Phi$ as that in [11], (79) will reduce to

$$\begin{aligned}\delta_\epsilon\Psi &= D_a\Phi\Gamma_a(\sin\theta\Gamma^6 + \cos\theta\Gamma_{45} - \Gamma_{04})\epsilon \\ &= D_a\Phi\Gamma_a\Gamma_{04}(\sin\theta\Gamma_{04}\Gamma^6 + \cos\theta\Gamma_{05} - 1)\epsilon = 0.\end{aligned} \tag{82}$$

The solution becomes 1/2 BPS. Moreover, for this state, $F_{0i} = 0$, so $P_k = Q_E = 0$, $E = \sqrt{Q_{M4}^2 + P_5^2}$. It may describe the monopole string extending along x_4 carrying the uniformly distributed $D0$ charge. Conversely, if $F_{a4} = \sin\theta D_a\Phi$, $D_aX^6 = -\cos\theta D_a\Phi$, $F_{ab} = 0$, $E = \sqrt{Q_E^2 + P_4^2}$. The solution describes the $F1$ string carrying P_4 momentum, which is also 1/2 BPS.

Another special kind of 1/2 BPS states have $F_{i4} = 0$ or $X^6 = 0$. When $X^6 = \theta = 0$, we get the instanton equation

$$F_{0i} = 0, \qquad \frac{1}{2}\varepsilon_{ilm4}F_{lm} = F_{i4}, \tag{83}$$

the solution of which describes the $D0$ branes revolved in $D4$ branes. $E = |P_5|$. The quantization of the instanton state gives the $5d$ massive $(2,0)$ tensor multiplet T_5 without charge. When $\theta \neq 0$,

$$F_{0i} = \sin\theta F_{i4}, \qquad \frac{1}{2}\varepsilon_{ilm4}F_{lm} = \cos\theta F_{i4}. \tag{84}$$

The $SO(4)$ symmetry is broken to $SO(3)$. Therefore, we may look for solutions which are translation invariant along x_4. $E = \sqrt{P_4^2 + P_5^2}$. The solution describes the $D0$ branes localized in R^3 carrying momentum P_4, which, in $D3$ picture, is the (p,q) strings winding x'^4. The quantization gives the $4d$ massive vector multiplet V_4 that is also the KK mode of the $5d$ massive tensor multiplet T_5. The original four position moduli of the instantons become the three position moduli plus one momentum moduli. $\tan\theta = P_4/P_5$. The (p,q) string can be open or closed, thus carries the $[i,j]$ charge or not, so is the corresponding $4d$ vector multiplet.

On the other hand, if $F_{i4} = \theta = 0$, the equations will be

$$F_{0i} = -D_iX^6, \qquad \frac{1}{2}\varepsilon_{ilm4}F_{lm} = 0, \tag{85}$$

whose solutions are $[i,j]$ $F1$ strings, the quantization of which gives the $5d$ vector multiplet V_5. $E = |Q_E|$. When $\theta \neq 0$,

$$F_{0i} = -\cos\theta D_iX^6, \qquad \frac{1}{2}\varepsilon_{ilm4}F_{lm} = \sin\theta D_iX^6. \tag{86}$$

The solution describes the bound state of the $[i,j]$ $F1$ and the $[i,j]$ monopole string extending along x_4, whose quantization also gives the $4d$ vector multiplet V_4. $E = \sqrt{Q_E^2 + Q_{M4}^2}$. $\tan\theta = Q_{M4}/Q_E$. In this case, θ is just the previously mentioned label for the selfdual strings parallel to the 45 plane. A reduction along x_5 is made to get the states with $P_5 = 0$. Selfdual strings extending along x_5 already have $P_5 = 0$ and is projected to a point in $5d$. The rest selfdual strings are projected to a straight line extending along x_4, which is the bound state of the $[i,j]$ $F1$ and the $[i,j]$ monopole string. $F1$ has the definite momentum $P_4 = 0$, while the monopole string carries no $D0$ charge, so the bound state is the zero mode of the $6d$ theory on $x_4 \times x_5$, which should be unique, but is now degenerate.

For 1/4 BPS state, when $\theta = 0$, we get (75), whose solution is the dyonic instanton, the quantization of which gives the $5d$ massive $(2,1)$ multiplet with 2^6 complex states composed by 1 spin-3/2 fermion, 13 spin-1/2 fermions, 2 selfdual tensors, 4 vectors and 10 scalars [11], which is actually the previously mentioned $T_5 \otimes ([1/2] \oplus [0] \oplus [0])$. When $\theta \neq 0$, the equations are (72) and (73). The solution corresponds to the bound state of the string and the monopole string, carrying the P_4 P_5 transverse momentum respectively. The string and the monopole string carry the different charge, for example, $[i,j]$ and $[j,k]$. The quantization gives the $4d$ $V_4 \otimes ([1/2] \oplus [0] \oplus [0])$ multiplet with 2^6 real states composed by 1 spin-3/2 fermion, 14 spin-1/2 fermions, 6 vectors and 14 scalars, which is the massive KK mode of $T_5 \otimes ([1/2] \oplus [0] \oplus [0])$. Notice that for the $F1$-$D0$ bound state, $D0$ is chargeless, so the corresponding multiplet can only carry the $[i,j]$ charge. On the other hand, for the $F1$-$D2$ bound state with the transverse momentum involved, $F1$ and $D2$ may carry the $[i,j]$ and $[j,k]$ charges, and so the corresponding multiplet may have the index $[i,j,k]$. Just as the 1/2 BPS case, $D0$ in momentum other than position eigenstate of x_4 can carry charge.

5.2 Selfdual string carrying the longitudinal momentum

It is convenient to work in $D3$ picture. With x_4 compactified, under the T-duality transformation along x_4, $A_4 \to X^4$. Let $F_{0a} = E_a$, $\frac{1}{2}\epsilon_{abc4}F_{bc} = B_a$, (72) and (73) could be rewritten as

$$E_a = \sin\theta D_a X^4 - \cos\theta D_a X^6, \tag{87}$$

$$B_a = \cos\theta D_a X^4 + \sin\theta D_a X^6, \tag{88}$$

which are the standard BPS equations for the $\mathcal{N} = 4$ SYM theory with two scalar fields X^4 and X^6 turned on. In the language of the $\mathcal{N} = 4$ $SU(N)$ SYM theory,

$$\int dS_a\, E_a = e\mathbf{p}\cdot\mathbf{H}, \quad \int dS_a\, B_a = \frac{4\pi}{e}\mathbf{q}\cdot\mathbf{H}, \tag{89}$$

where the vectors $\mathbf{p}$ and $\mathbf{q}$ are the electric and the magnetic charges respectively. $\mathbf{H}$ generates the Cartan subalgbra of $SU(N)$.

$$\mathbf{p}\cdot\mathbf{H} = \mathrm{diag}(p_1, p_2, \cdots, p_N), \quad \mathbf{q}\cdot\mathbf{H} = \mathrm{diag}(q_1, q_2, \cdots, q_N), \tag{90}$$

$\sum_{i=1}^N p_i = \sum_{i=1}^N q_i = 0$. Suppose $\left\langle X^I \right\rangle = \mathbf{v}^I \cdot \mathbf{H} = \mathrm{diag}(v_1^I, v_2^I, \cdots, v_N^I)$, $v_1^6 \geq v_2^6 \geq \cdots \geq v_N^6$,

$$\mathcal{Q}_E^4 = \int d^3x\ \partial_a \mathrm{tr}\Big[X^4 E_a\Big] = e\mathbf{p}\cdot\mathbf{v}^4 = e\sum_{i=1}^N p_i v_i^4 \sim -P_4, \tag{91}$$

$$\mathcal{Q}_E^6 = \int d^3x\ \partial_a \mathrm{tr}\Big[X^6 E_a\Big] = e\mathbf{p}\cdot\mathbf{v}^6 = e\sum_{i=1}^N p_i v_i^6 \sim Q_E, \tag{92}$$

$$\mathcal{Q}_M^4 = \int d^3x\ \partial_a \mathrm{tr}\Big[X^4 B_a\Big] = \frac{4\pi}{e}\mathbf{q}\cdot\mathbf{v}^4 = \frac{4\pi}{e}\sum_{i=1}^N q_i v_i^4 \sim -P_5, \tag{93}$$

$$\mathcal{Q}_M^6 = \int d^3x\ \partial_a \mathrm{tr}\Big[X^6 B_a\Big] = \frac{4\pi}{e}\mathbf{q}\cdot\mathbf{v}^6 = \frac{4\pi}{e}\sum_{i=1}^N q_i v_i^6 \sim -Q_{M4}. \tag{94}$$

The energy becomes[9]

$$E = \sqrt{(\mathcal{Q}_E^4 + \mathcal{Q}_M^6)^2 + (\mathcal{Q}_E^6 - \mathcal{Q}_M^4)^2}. \tag{95}$$

$x^4 \sim x^4 + 2\pi n R$. The transverse position of the i_{th} $D3$ brane in 4-6 plane could be denoted by (v_i^4, v_i^6), where $v_i^I \in (-\infty, +\infty)$.

The generic 1/2 BPS state is the (p,q) string connecting the i j $D3$ branes with the mass

$$E = \sqrt{[ep(v_i^6 - v_j^6) - \frac{4\pi}{e}q(v_i^4 - v_j^4)]^2 + [\frac{4\pi}{e}q(v_i^6 - v_j^6) + ep(v_i^4 - v_j^4)]^2}. \tag{96}$$

Especially, if $e^2 p(v_i^6 - v_j^6) = 4\pi q(v_i^4 - v_j^4)$, the state will reduce to a $[j,i]$ $D2$ brane carrying $[j,i]$ P_4 momentum, while if $4\pi q(v_i^6 - v_j^6) = -e^2 p(v_i^4 - v_j^4)$, the state will become a $[i,j]$ string with $[j,i]$ $D0$ charge. Notice that in this case, the P_4 momentum and the $D0$ charge spread uniformly over the $D2$ branes and the strings.

The simplest 1/4 BPS state is the 3-string junction with i j k representing three distinct $D3$ branes with coordinates (v_i^4, v_i^6), (v_j^4, v_j^6), (v_k^4, v_k^6). With the charge vector $\mathbf{v}_e = (1,0,-1)$, $\mathbf{v}_m = (0,1,-1)$, the mass is

$$E = \sqrt{[e(v_i^6 - v_k^6) - \frac{4\pi}{e}(v_j^4 - v_k^4)]^2 + [\frac{4\pi}{e}(v_j^6 - v_k^6) + e(v_i^4 - v_k^4)]^2}. \tag{97}$$

The corresponding state on $D4$ is a $[i,k]$ string with $v_k^4 - v_j^4$ $D0$ charge and a $[k,j]$ $D2$ brane with $v_k^4 - v_i^4$ P_4 momentum. Especially, when $e^2(v_i^6 - v_k^6) = 4\pi(v_j^4 - v_k^4)$, the state reduces to the $[k,j]$ $D2$ brane carrying $[k,i]$ P_4 momentum, while when $4\pi(v_j^6 - v_k^6) = e^2(v_k^4 - v_i^4)$, the state reduces to the $[i,k]$ string carrying $[k,j]$ $D0$ charge. The $[i,k]$ string and the $[k,j]$ $D2$ brane are parallel, so the bound state does not exist, nevertheless, with suitable amount of P_4 momentum and the $D0$ charge, the bound state may form. With the given A_4, the $[i,j]$ string ($D2$ brane) can only carry the $[i,j]$ P_4 momentum ($D0$ charge). However, they can carry the $[j,k]$ or $[i,k]$ $D0$ charge (P_4 momentum), which is actually the transverse

[9]For (95) to be valid, for the given $\mathbf{v}^I$, $\mathbf{p}$ $\mathbf{q}$ should be selected so that $Z_+ \geq Z_-$, otherwise $E = \sqrt{(\mathcal{Q}_E^4 - \mathcal{Q}_M^6)^2 + (\mathcal{Q}_E^6 + \mathcal{Q}_M^4)^2}$.

momentum of the $[j,k]$ or $[i,k]$ $D2$ brane (string). The $[i,j]$ selfdual string carrying the $[j,k]$ longitudinal momentum has the N^3 scaling.

Now, consider string webs with more external legs. For $i \geq k \geq l \geq j$, the $[i,j]$ $D2$ brane (string) $[k,l]$ string ($D2$ brane) bound states do not exist. The bound state may exist if the $[k,l]$ string ($D2$ brane) carries the appropriate P_4 momentum ($D0$ charge). In general, the charge vector can be taken as $p_i = 1$, $p_j = -1$, $p_a = 0$ for $a \neq i,j$, $q_a = 0$ for $a > i$ or $a < j$.

$$\begin{aligned} P_4 &= e(v_j^4 - v_i^4), \quad Q_E = e(v_i^6 - v_j^6), \\ P_5 &= -\frac{4\pi}{e}\sum_{m=j}^{i} q_m v_m^4 = \frac{4\pi}{e}\sum_{m=j}^{i} r_m (v_{m+1}^4 - v_m^4), \\ Q_{M4} &= -\frac{4\pi}{e}\sum_{m=j}^{i} q_m v_m^6 = \frac{4\pi}{e}\sum_{m=j}^{i} r_m (v_{m+1}^6 - v_m^6). \end{aligned} \tag{98}$$

This is the bound state of the $[i,j]$ string and r_m $[m+1,m]$ $D2$ branes each carrying $v_{m+1}^4 - v_m^4$ $[m+1,m]$ $D0$ charge. Especially, if the $[i,j]$ string carries the transverse momentum P_4 so that $P_4 + Q_{M4} = 0$, the state will reduce to the $[i,j]$ string with r_m $v_{m+1}^4 - v_m^4$ $[m+1,m]$ $D0$ charge. Conversely, one may let $q_i = 1$, $q_j = -1$, $q_a = 0$ for $a \neq i,j$, $p_a = 0$ for $a > i$ or $a < j$.

$$\begin{aligned} P_5 &= \frac{4\pi}{e}(v_j^4 - v_i^4), \quad Q_{M4} = \frac{4\pi}{e}(v_j^6 - v_i^6) \\ P_4 &= -e\sum_{m=j}^{i} p_m v_m^4 = e\sum_{m=j}^{i} s_m (v_{m+1}^4 - v_m^4), \\ Q_E &= e\sum_{m=j}^{i} p_m v_m^6 = -e\sum_{m=j}^{i} s_m (v_{m+1}^6 - v_m^6). \end{aligned} \tag{99}$$

The corresponding state is the bound state of the $[j,i]$ $D2$ brane and s_m $[m,m+1]$ strings each carrying the $v_{m+1}^4 - v_m^4$ P_4 momentum. When $P_5 + Q_E = 0$, the state becomes $[j,i]$ $D2$ brane carrying s_m $v_{m+1}^4 - v_m^4$ $[m+1,m]$ P_4 momentum.

We can give a more precise description for these longitudinal-momentum-carrying states. For example, for string web in Fig.1, suppose the strings extending in x_4 x_6 directions are $(1,0)$ and $(0,1)$ strings, while the rest ones are $(1,1)$ strings, then the state could be taken as the $[i,n]$ $D2$ extending along $x_4 \times x_6$, for which, the $[i,j]$ $[k,l]$ $[m,n]$ $D2$ carry the zero P_4 momentum, the $[j,k]$ $[l,m]$ $D2$ have the uniformly distributed $T_{F1}|v_{ab}^4|$, $T_{F1}|v_{cd}^4|$, P_4 momentum, while the rest $T_{F1}|v_{ja}^4|$, $T_{F1}|v_{bk}^4|$, $T_{F1}|v_{lc}^4|$, $T_{F1}|v_{dm}^4|$, P_4 momentums are localized on the j_{th}, k_{th}, l_{th}, m_{th} $D4$ branes.

$[i,j]$ $D2$ ($F1$) is composed by $[i,i+1] \cdots [j-1,j]$ $D2$'s ($F1$'s). Each $[a,a+1]$ $D2$ ($F1$) must have the same transverse velocity, otherwise, the bound state cannot be formed. On the other hand, the longitudinal momentums along x_4 (x_5) on each $[a,a+1]$ $D2$ ($F1$) are independent, so the degrees of freedom on the $[i,j]$ $D2$ ($F1$) is $j-i$. Altogether, there are $N(N-1)/2$ $[i,j]$ $D2$ ($F1$), therefore, the total number of degrees of freedom is $(N^3-N)/6$.

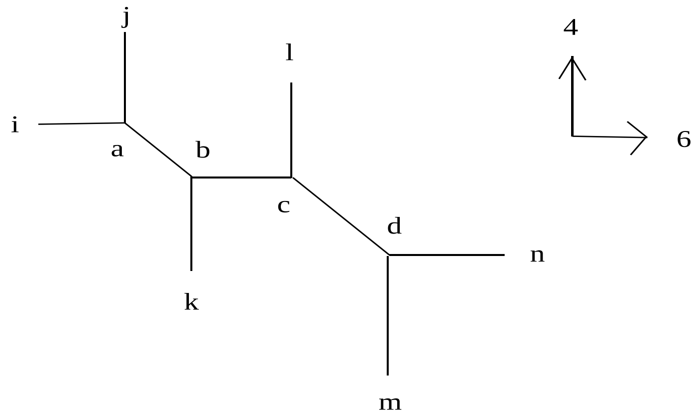

Figure 1: The $ijklmn$ string web

The N^3 scaling comes from the longitudinal momentum. Both transverse momentum and the longitudinal momentum carry the charge. The $[i, j]$ $D2$ $(F1)$ can only carry the $[i, j]$ transverse momentum but the $[k, l]$ longitudinal momentum for any $i \le k < l \le j$. The index calculation in [28] also showed the $(N^3 - N)/6$ degrees of freedom for the longitudinal momentum mode on open $D2$'s connecting $D4$'s.

6 The degrees of freedom at the triple intersection of $M5$ branes

In this section, we will consider the the triple intersecting configuration of the $M5$ branes $5\perp5\perp5$. We will discuss the possible string junctions and their relevance with the N^3 degrees of freedom at the triple interaction.

Suppose there are N_1, N_2, N_3, $M5$ branes extending in 0 1 2 3 4 5 direction, 0 1 2 3 6 7 direction, and 0 1 4 5 6 7 direction respectively (see Fig.2). The common transverse space

	0	1	2	3	4	5	6	7	8	9	10
M5	*	*	*	*	*	*					
M5	*	*	*	*			*	*			
M5	*	*			*	*	*	*			

Figure 2: The $M5$ $M5$ $M5$ configuration

is x_8 x_9 x_{10}, while the common longitudinal spacetime is x_0 x_1 with x_0 the time direction.

If $x_8 = x_9 = x_{10}$, the $N_1 + N_2 + N_3$ $M5$ branes will have $N_1 N_2 N_3$ triple intersections no matter whether each bunch of $M5$ branes are coincident or not. The black hole entropy calculation shows that there are $N_1 N_2 N_3$ degrees of freedom at the triple intersections, so each intersection will offer one degree of freedom [29]. The situation can be compared with the $4\perp 4$ configuration for N_1 and N_2 intersecting $D4$ branes with $N_1 N_2$ $3d$ intersections. There are $U(1) \times U(1)$ massless hypermultiplets living at each intersection, producing the $N_1 N_2$ entropy. So, we may expect that similarly the triple intersection will also capture some nonabelian features of $M5$.

Consider one intersection and label the three $M5$ branes by i, j, k. In the most generic case, i j k $M5$ branes appear as three points v_i^I v_j^I v_k^I in $x_8 \times x_9 \times x_{10}$ transverse space. Still, we want to compactify two longitudinal dimensions of $M5$ branes to simplify the problem. There are two distinct possibilities: $x_2 \times x_4$ and $x_2 \times x_1$. M theory compactified on x_2 gives the type IIA string theory, with the i j k $M5$'s becoming the $D4$ $D4$ $NS5$. The triple intersection of the $D4$ $D4$ $NS5$ branes still have the $N_1 N_2 N_3$ entropy, so the KK mode along x_2 can be safely dropped[10].

	0	1	2	3	4	5	6	7	8	9	10
D3	*	*		*		*					
D5	*	*		*	*		*	*			
NS5	*	*			*	*	*	*			

Figure 3: The $D3$ $D5$ $NS5$ configuration

Then compactify on x_4 with the radius R_4 and do a T-duality transformation, we get $D3$ $D5$ $NS5$ (see Fig.3). The state carrying $[i, j, k]$ index is the 3-string junction with (p, q), $(p, 0)$, $(0, q)$ strings ending on $D3$ $D5$ $NS5$, which will become massless when $v_i^I = v_j^I = v_k^I$. This is the scenario discussed in [30]. The 3-string junction is the point-like particle in $x_0 \times x_1$, so they may give the field $f^{ijk}(x_0, x_1)$ localized at the intersection. In M theory with $x_2 \times x_4$ compactified to T^2, the 3-string junction is lifted to a $M2$ embedded along a holomorphic curve in $x_2 \times x_4 \times x_8 \times x_9 \times x_{10}$, ending on the three $M5$'s along (pR_2, qR_4), pR_2, qR_4 [26]. Still, the problem is that when the three $M5$ branes intersect, the 3-string junction is at the threshold and may decay into the component strings. If they do decay, then at the triple intersection, there will be no BPS state related with all three branes.

The other possibility is to compactify on x_1 with the radius R_1 and also do a T-duality transformation. We get $D3$ $D3$ $NS5$ (see Fig.4). In $x_8 \times x_9 \times x_{10}$, no string junction can

[10]The P_2 momentum may have the relevance with the $c = 6$ central charge. With one $M5$ fixed, there are 4 moduli to characterize the $5\perp 5\perp 5$ intersection, while for $D4$ $D4$ $NS5$, only 3 moduli are left, since the motion along x_2 is frozen.

	0	1	2	3	4	5	6	7	8	9	10
D3	*			*	*	*					
D3	*			*			*	*			
NS5	*	*			*	*	*	*			

Figure 4: The $D3$ $D3$ $NS5$ configuration

be formed. We may consider the 3-string junction in, for example, $x_1 \times x_8$ plane. The KK mode along x_1 cannot be dropped. Actually, in the T-dual picture, x_1 is a circle with the radius $\frac{1}{T_{F1}R_1}$, so $D3$ and $D3$ will be separated with the distance $\frac{m}{T_{F1}R_1}$ in the covering space.

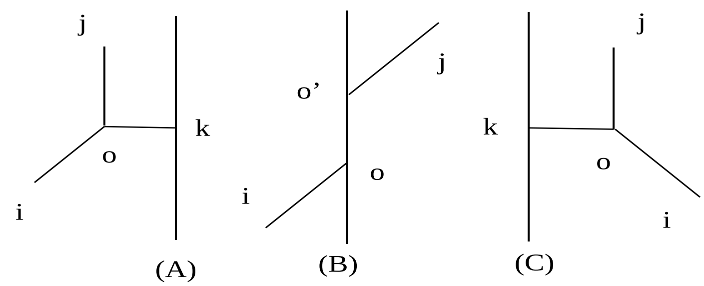

Figure 5: The 3-string junction in 18 plane

In $x_1 \times x_8$ plane, $NS5$ is a straight line locating in v_k^8, while the i j $D3$'s appear as two points with coordinates (v_i^1, v_i^8), (v_j^1, v_j^8), $v_{ij}^1 = \frac{m}{T_{F1}R_1}$. The simplest 3-string junctions are given in Fig.5. In Figure 5 (A) and (C), the io oj ok strings carry the charge (p,q) $(p,0)$ $(0,q)$, $\tan\angle ioj = -\frac{qR_1}{pR_2}$. In Figure 5 (B), the io oo' $o'j$ strings carry the charge (p,q) $(p,0)$ (p,q), $\tan\angle ioo' = \tan\angle oo'j = -\frac{qR_1}{pR_2}$. Actually, there are also $(0,q)$ oa string and the $(0,q)$ bo' string ending on $NS5$ with the zero length. The string junctions like this always exist. The mass of the string junctions in (A) and (C) is $qT_{D1}|v_{ik}^8| + pT_{F1}|v_{ij}^1|$. The mass of the string junctions in (B) is $qT_{D1}|v_{ij}^8| + pT_{F1}|v_{ij}^1|$. In the T-dual picture, Figure 5 (A) corresponds to q $[i,k]$ monopole strings with tension $T_{M2}|v_{ik}^8|$ wrapping x_1 carrying the $[i,j]$ longitudinal momentum $P_1 = pm/R_1$. The situation is similar for Figure 5 (C). Figure 5 (B) corresponds to q $[i,j]$ monopole strings with tension $T_{M2}|v_{ij}^8|$ wrapping x_1 carrying the $[i,j]$ longitudinal momentum $P_1 = pm/R_1$.

When $v_i^8 = v_j^8 = v_k^8$, Figure 5 (A) and (C) reduce to the $[i,k]$ tensionless monopole strings wrapping x_1 carrying the $[i,j]$ longitudinal momentum $P_1 = m/R_1$, which is offered by the potentially existing $[i,j]$ massless string. In Figure 5 (B), with oa oo' or oo' bo' kept, the state becomes the $[i,k]$ or $[k,j]$ tensionless monopole string wrapping x_1 carrying the $[i,j]$

longitudinal momentum $P_1 = m/R_1$ offered by the $[i, j]$ massless string. Monopole string wrapping x_1 carrying P_1 momentum corresponds to the KK mode of $f^{ijk}(x_0, x_1)$ along x_1. Again, at the intersection, the state may decay into the monopole string and the string, and then there will be no BPS state relevant to all three branes.

At the $5\perp5\perp5$ intersection of three $NS5_A$'s or $NS5_B$'s or $D5$'s, there are type IIA strings, or type IIB strings, or D-strings living at the intersection. The F-string or D-string has the $4d$ transverse monition thus may produce the $c = 6$ central charge. The oscillation of the F-string or D-string gives the P_1 momentum. The problem is that neither F-string nor D-string carries charge, so it is difficult to explain their relation with the three intersecting branes.

7 Discussion

In this paper, we considered the momentum modes of the $M5$ branes on a plane, which are the transverse momentum of the selfdual strings parallel to that plane. Different from the D branes, on which, the momentum modes are carried by the same kind of point-like excitations, here, the unparallel momentum modes are carried by selfdual strings with the different orientations. Selfdual strings with the same orientation gives a $5d$ SYM theory with the field configurations taking the zero Pontryagin number. The original $6d$ $(2, 0)$ tensor multiplet field is then decomposed into a series of θ-parameterized $5d$ $U(N)$ SYM fields, among which, fields labeled by the same θ have the standard SYM-type interaction. Fields labeled by different θ are associated with the selfdual strings with the different orientations. As a result, the $[i, j] + [j, k] \to [i, k]$ relation is not valid and the coupling cannot be realized as the standard matrix multiplication.

Since the bound state of the $[i, j]$ θ_1 selfdual string and the $[j, k]$ θ_2 selfdual string is not some $[i, k]$ selfdual string but the 3-string junction, we may also include the string junction into the theory. Each 3-string junction is characterized by $(\theta_1, \theta_2, \theta_3)$, forming the tri-fundamental or anti-tri-fundamental representation of $U(N)$, and may couple with the θ_1 θ_2 θ_3 selfdual strings in adjoint representation of $U(N)$. $[i, l] + [l, j, k] \to [i, j, k]$, $[j, m] + [i, m, k] \to [i, j, k]$, $[k, n] + [i, j, n] \to [i, j, k]$.

The quantization of the 3-string junction will give the higher-spin multiplet, for which the simplest one is the $(2, 1)$ multiplet with the highest spin $3/2$. It is unclear whether the introducing of the 3-string junction will solve the problem or bring more problems, since at the beginning, we only want to get a theory for the $(2, 0)$ tensor multiplet. The incorporation of the $5d$ massive $(2, 1)$ multiplet into the $6d$ $(2, 0)$ theory compactified on S^1 was also discussed in [15], where it was suggested that the algebraic structure of the $6d$ $(2, 0)$ theory may have a fermionic symmetry in addition to the self-dual tensor gauge symmetry.

Each 3-string junction carries three indices, so they may offer the N^3 degrees of freedom on N $M5$ branes. However, the existing of the 3-string junction is severely restricted by the marginal stability curve, outside of which, the string junction may decay into the strings.

For the given vacuum expectation values of the scalar fields on $M5$, the momentum of the string junction on that plane cannot be arbitrary. Especially, on coincident $M5$ branes, the 3-string junctions are at best at the marginal stability curve, so it is quite likely that they may decay.

Maybe it is easier consider the problem in the dual $D3$ picture. For $D3$ with the transverse dimension x'^{45} compactified, the winding mode of the (p,q) strings is dual to the $(n/R_4, m/R_5)$ momentum mode on $M5$. For the give p and q, open (p,q) strings with the arbitrary winding numbers have the SYM interaction. Then the questions are whether the open (p,q) (r,s) strings can interact or not, if can, in which way, what is the situation when $D3$ branes are coincident.

Among all selfdual strings, only those parallel to a given plane are taken as the perturbative degrees of freedom; nevertheless, different planes give the dual theories. One may compare the $6d$ theory with the $5d$ and $4d$ theories coming from the reductions on x_5 and $x_4 \times x_5$. Obviously, selfdual string extending along x_5 is the only candidate to define the $5d$ theory. However, for $4d$ theory, any selfdual string parallel to the 45 plane, carrying zero transverse momentum along it can act as the perturbative degrees of freedom. Only one is selected to give the $4d$ field, while the rest ones define the dual theories. Similarly, for $6d$ theory, selfdual strings parallel to a specific plane can be taken to give the $6d$ field, while the other planes give the dual versions. $M5$ on $S_1 \times S_2 \times S_3 \times S_4 \times S_5$ is dual to $D3$ on $S_i \times S_j \times S_k$ with a transverse S'^{lm}, where $\{1,2,3,4,5\} = \{i,j,k,l,m\}$. The (p,q) string ending on $D3$ winding S'^{lm} is dual to the selfdual string extending in (qR_l, pR_m) direction, carrying transverse momentum in $x_l \times x_m$, localized in $x_i \times x_j \times x_k$. There are 10 possible dual theories, corresponding to choosing the selfdual strings parallel to 10 different $2d$ subspaces.

$M5$ on T^5 is $SL(5,Z)$ invariant. However, the $6d$ theory on $M5$ does not have the explicit $SL(5,Z)$ invariance. The $SL(5,Z)$ U-duality transformation, or the $SO(5)$ U-duality transformation in R^5, is not a simple differorphism transformation but is also accompanied by a reallocation of the perturbative and the nonpertubative degrees of freedom. The U-dual $6d$ theories are equivalent, so the $SL(5,Z)$ transformation is just like a change of the gauge. This is similar with the $D3$. Although $D3$ is S-duality invariant, the $4d$ theory on $D3$ does not have the $SL(2,Z)$ invariance. The nonpertubative $SL(2,Z)$ transformation gives the equivalent $4d$ theories.

Acknowledgments: The work is supported by the Mitchell-Heep Chair in High Energy Physics and by the DOE grant DE-FG03-95-Er-40917.

References

[1] E. Witten, “String theory dynamics in various dimensions”, Nucl. Phys. B443, 85-126 (1995), hep-th/9503124.

[2] A. Strominger, “Open p-branes”, Phys. Lett. B383, 44-47 (1996), hep-th/9512059.

[3] K. Becker and M. Becker, “Boundaries in M-theory”, Nucl. Phys. B472, 221-230 (1996), hep-th/9602071.

[4] R. Dijkgraaf, E. Verlinde and H. Verlinde, “BPS spectrum of the five-brane and black hole entropy”, Nucl. Phys. B486, 77-88 (1997), hep-th/9603126.

[5] R. Dijkgraaf, E. Verlinde and H. Verlinde, “BPS quantization of the five-brane”, Nucl. Phys. B486, 89-113 (1997), hep-th/9604055.

[6] N. Seiberg, “New Theories in Six Dimensions and Matrix Description of M-theory on T^5 and T^5/Z_2”, Phys. Lett. B408 (1997) 98, hep-th/9705221.

[7] E. Witten, “Conformal Field Theory In Four And Six Dimensions”, [arXiv:0712.0157 [math.RT]].

[8] E. Witten, “Geometric Langlands From Six Dimensions”, [arXiv:0905.2720 [hep-th]].

[9] N. Lambert and C. Papageorgakis, “Nonabelian (2,0) tensor multiplets and 3-algebras”, JHEP 08 (2010) 083, [arXiv:1007.2982 [hep-th]].

[10] M. R. Douglas, “On D=5 super Yang-Mills theory and (2,0) theory”, JHEP 1102, 011 (2011), [arXiv:1012.2880 [hep-th]].

[11] N. Lambert, C. Papageorgakis, and M. Schmidt-Sommerfeld, “M5-branes, D4-branes and quantum 5D super-Yang-Mills”, JHEP 1101 (2011) 083, [arXiv:1012.2882 [hep-th]].

[12] O. Bergman, “Three-pronged strings and 1/4 BPS states in N = 4 super-Yang-Mills theory”, Nucl. Phys. B 525 (1998) 104, hep-th/9712211.

[13] K. M. Lee and P. Yi, ”Dyons in N = 4 supersymmetric theories and three-pronged strings”, Phys. Rev. D 58, 066005 (1998), hep-th/9804174.

[14] D. Bak, K. M. Lee and P. Yi, “Quantum 1/4 BPS dyons”, Phys. Rev. D 61, 045003 (2000), hep-th/9907090.

[15] B. Czech, Y.-t. Huang, and M. Rozali, “Amplitudes for Multiple M5 Branes”, [arXiv:1110.2791 [hep-th]].

[16] O. Bergman and B. Kol, “String webs and 1/4 BPS monopoles”, Nucl. Phys. B 536 (1998) 149, hep-th/9804160.

[17] M. Berkooz, M. Rozali and N. Seiberg, “Matrix Description of M-theory on T^4 and T^5”, Phys. Lett. B408 (1997) 105, hep-th/9704089.

[18] O. Aharony, “A brief review of little string theories”, Class. Quant. Grav. 17: 929-938, (2000), hep-th/9911147.

[19] C. Hull, “BPS supermultiplets in five-dimensions”, JHEP 0006 (2000) 019, hep-th/0004086.

[20] M. Abou-Zeid, B. de Wit, D. Lust and H. Nicolai, “Space-Time Supersymmetry, IIA/B Duality and M-Theory”, Phys. Lett. B466 (1999) 144, hep-th/9908169.

[21] K. Lee and P. Yi, “Monopoles and instantons on partially compactified D branes”, Phys. Rev. D 56 (1997) 3711, hep-th/9702107.

[22] K. Lee, “Instantons and magnetic monopoles on $R^3 \times S^1$ with arbitrary simple gauge groups”, Phys. Lett. B 426 (1998) 323, hep-th/9802012.

[23] S. Bolognesi and K. M. Lee, “1/4 BPS string junctions and N^3 problem in 6-dim $(2,0)$ superconformal theories”, Phys. Rev. D 84 (2011) 126018, [arXiv:1105.5073 [hep-th]].

[24] D. Gaiotto, “N=2 Dualities”, [arXiv:0904.2715 [hep-th]].

[25] D. Gaiotto and J. Maldacena, “The Gravity Duals of N = 2 Superconformal Field Theories”, [arXiv:0904.4466 [hep-th]].

[26] M. Krogh and S. Lee, “String network from M theory”, Nucl. Phys. B516 (1998) 241-254, hep-th/9712050.

[27] D. Young, “Wilson Loops in Five-Dimensional Super-Yang-Mills”, JHEP 052 (2012), [arXiv:1112.3309 [hep-th]].

[28] H.-C. Kim, S. Kim, E. Koh, K. Lee and S. Lee, “On instantons as Kaluza-Klein modes of M5-branes”, [arXiv:1110.2175 [hep-th]].

[29] I. R. Klebanov and A. A. Tseytlin, “Intersecting M-branes as four-dimensional black holes”, Nucl. Phys. B475 (1996) 179-192, hep-th/9604166.

[30] D. Berenstein and R. Leigh, “String junctions and bound states of intersecting branes”, Phys. Rev. D60 (1999) 026005, hep-th/9812142.